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TRANSVERSE IMPACT ON BEAMS AND PLATES WITH ARBITRARY EDGE CONDITIONS

by

A. Cemal Eringen

to

Office of Naval Research
Department of the Navy
Contract No. N7onr-32909

Department of Mechanics
Illinois Institute of Technology
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a - longer side of rectangular plate; outer radius of circular plate

a_1, a_2, a_3 - constants - Equation (44)

a_{mn} - constants - Equations (17), (18)

A - area of plate or total length of beam

$a, a_2, A_i(\lambda_{mn}), A'(\lambda_{mn}), \bar{A}_i, B_i(\lambda_{mn}), B_{ij}(\lambda_{mn})$ - functions connected with Galerkin Method - Equations (46)-(53)

b - shorter side of rectangular plate

b_{mn}, c_{mn} - constants - Equations (19)-(23)

B_{mn}, C_{mn} - constants - Equation (64)

$d, d', D(\lambda_{mn}), D'(\lambda_{mn})$ - functions connected with Galerkin Method - Equations (46)-(53)

D - bending rigidity - Equation (5)

E - Young's modulus; common moduli of elasticity of sphere and plate

F_i - constants - Equation (61)

$F(t)$ - contact force

g - acceleration due to gravity

g_{mn}, g_n - eigenfunctions - Equations (17), (18)

G - Equation (12)

h - thickness of plate or beam

J_n, I_n - Bessel functions of first kind, with real and imaginary arguments respectively

$K(x)$ - kernel of the integral equation

k - Hertz constant

L - length of beam

$L(t)$ - error function - Equation (40)

M - total mass of plate or beam

m - mass of striking body

m,n - integers

M_n - bending moment

M_x, M_y - bending moments per unit length

M_{xy} - twisting moments per unit length

n - external normal to the boundary curve

p_{mn} - eigenvalues - Equations (17),(18)

q - load per unit area or unit length

Q_x, Q_y - transverse shear forces per unit length

r_s - radius of sphere

$R_1, R_2, R_1', R_2', R_1'', R_2''$ - auxiliary functions used in Collocation Method - Equations (57)-(60)

$S(x)$ - function - Equation (54)

t - time

T_c - contact time

v_o - striking velocity

V_n - transverse shear resultant

$w(x,y,t)$ - deflection

w_o - deflection at point of contact

w_s - displacement of sphere under action of force $F(t)$

$x = t/T_o$ - non-dimensional time - Equation (41)

$X(x), Y(x)$ - shape functions - Equations (54),(56)

α - relative approach of striking bodies

α_{mn} - circular frequency - Equation (15)

$\alpha_i, \alpha_i', \beta_{ij}, \gamma, \delta, \delta', \eta_i$ - functions connected with Galerkin Method - Equations (46)-(53)

$\Gamma(u)$ - Gamma function

$\delta(x)$ - Dirac-delta function

Δ - Laplacian operator

λ_{mn} - constants- Equation (50)

μ - mass ratio m/M

ν - Poisson's ratio

ρ - weight density per unit length or unit area

$\rho_1(\lambda_{mn}), \rho_2(\lambda_{mn}), \rho_1'(\lambda_{mn}), \rho_2'(\lambda_{mn})$ - Auxiliary functions used in Collocation Method - Equations (57)-(60)

σ_x, σ_y - flexural stresses

τ_{xz}, τ_{yz} - shear stresses

$\phi(x)$ - Equation (56)

ω_{mn} - Equation (41)

Subscripts after a comma represent differentiation, ie., $\partial u/\partial x = u_{,x}$

ABSTRACT

Flexural deflections of several plates and beams under an unknown transverse, concentrated, time dependent force are solved for various edge conditions. The consideration of displacements and the use of Hertz's law of impact at the point of contact lead to a non-linear integral equation for the contact force in all cases of transverse impact. Two methods which are developed in the previous report are then used to treat this equation: (a) Generalized Galerkin Method; (b) Collocation Method. Formulas for the maximum contact force and contact time are obtained which apply to all elastic impact problems. With the aid of various curves given in this report calculation can be shortened to a minimum. Impacts on circular and rectangular plates are studied in detail.

1. Introduction

To determine the deflection and stresses in a beam or a plate struck transversely by an elastic body one must know the local compression - contact force relationship and the flexural deflection under an unknown concentrated time dependent contact force. A solution for the first is to use the Hertz law of impact [2], [3], and [4], which is known to have a wide range of applicability [5]. The flexural deflections, on the other hand, can be determined by using the classical theory of vibrations of beams and plates. Consideration of the displacement at the point of contact then leads to a non-linear integral equation for the contact force. Various authors studied the problem of central impact of a sphere on a simply supported beam previously. A discussion of various methods given by these authors is presented in [1].

The impact problems involving beams with different types of support conditions and plates with different shapes and non-central impact problems are as yet untouched, except for the paper by K. Karas [6]. He treats the problem of central impact on simply supported, rectangular plates using the method given by Lennertz and the step-by-step integration method used by Timoshenko for the beam problem. A criticism of these methods may be found in [1] also. Briefly, the method introduced by Lennertz oversimplifies the problem leading to large discrepancies between the correct and approximate contact forces, while the step-by-step integration method presents a tedious process and must be carried out for each individual problem.

In the present report, the problem of non-central impact on beams and plates having general edge conditions is formulated in such a way as to unify the transverse impact problems. Consequently, for the free transverse oscillation problem leading to normal modes, the integral equation of impact can be written very shortly. Explicit forms for the integral equations are given for the following cases:

(a) Beams:

1) Simply supported beams

2) Beams having both ends clamped

3) Cantilever beams

(b) Circular Plates:

1) Simply supported circular plates

2) Circular plates with clamped outer edge.

(c) Rectangular Plates:

1) Simply supported at all edges

2) Simply supported at two parallel edges and clamped at the others.

Two general methods are then used to obtain approximate solutions of the integral equations of contact force:

(a) Generalized Galerkin Method

(b) Collocation Method.

Discussion of these methods is given in [1].

Explicit formulas are given for the maximum contact force and for the contact time for various different types of shape functions. Plots are made for various functions to facilitate computation for a given problem. In particular, beams, circular plates and rectangular plates with simply supported edges are studied in detail.

2. Formulation of the Problem

A beam or a plate is struck transversely by a mass m having a spherical surface at the point of contact, and striking velocity v_o, (Fig. 1).

The problem is to determine: the contact force $F(t)$, deflection $w(x,y,t)$ and the flexural stresses.

Deflection and the stresses are functionals of the contact force $F(t)$. Consequently, $F(t)$ must be determined first.

The formulation of this problem can be effected only under certain assumptions, namely: (a) all assumptions of the classical theory of plates or beams are applicable; (b) the Hertz law of impact is valid. The last assumption states that:

$$\alpha = k\, F^{2/3}(t) \tag{1}$$

where α is the relative approach of striking bodies and k is the Hertz constant [1], [2], [3]. The relative approach is the difference between displacements of the plate and the striking body measured from the instant of initial contact (Fig. 1). Hence:

$$\alpha = w_s - w_o \tag{2}$$

where w_s is the displacement of the sphere under the action of the force $F(t)$ and w_o is the deflection of the plate at the point of contact. Here w_s is given by:

$$w_s = v_o t - \frac{1}{m} \int_o^t (t-\tau)\, F(\tau) d\tau \tag{3}$$

The deflection of the beam or the plate is in turn obtained by solving the following differential equation:

$$D\Delta^2 w + \frac{\rho}{g} \frac{\partial^2 w}{\partial t^2} = q(x,y,t) \tag{4}$$

$$D = Eh^3/12(1 - \nu^2) \quad , \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5}$$

Here E is Young's modulus, h is the thickness, ρ is the weight density per unit area, g is the acceleration due to gravity, q is the load per unit area. Equation (4) reduces to beam equation if one takes EI for D and $y = \partial/\partial y = 0$. In this case, ρ and q must be interpreted as the weight density and the load per unit length. In the following analysis beam and plate problems are interpreted as one. The difference is in the additional dimension y and the interpretation of D and ρ which can be adjusted easily to the method of solution. The resulting formulas, however, will be given separately for beams and plates.

The concentrated load $F(t)$ is obtained as a limiting process of a uniformly distributed load q over a small area e^2 having the shape of the plate boundary. In beams, q is distributed uniformly over a range e . Thus, by letting $q \rightarrow \infty$ while $e \rightarrow 0$, the contact force $F(t)$ is obtained:

$$F(t) = \lim_{\substack{e \rightarrow 0 \\ q \rightarrow \infty}} \int_{(A)} q \, \partial A \tag{6}$$

Hence equation (4) with q subject to (6) must be solved under initial and boundary conditions to obtain w_o . Once this is done, the force $F(t)$ can then be evaluated by solving the integral equation obtained by combining equations (1) to (3), namely:

$$kF^{2/3}(t) = v_o t - \frac{1}{m}\int_o^t (t-\tau)\ F(\tau)\ d\tau - w_o \tag{7}$$

3. Flexural Deflection

Deflection w of a plate or a beam is uniquely determined when equation (4) is solved under appropriate initial and boundary conditions. Initial conditions are:

$$w(x,y,0) = 0 \qquad \dot{w}(x,y,0) = 0 \tag{8}$$

where the dot represents differentiation with respect to time. Boundary conditions used in practice have great variety. These conditions consist of specifying any two of the following four quantities or their linear combinations on the boundary:

$$\text{Given:} \quad w \, , \quad \partial w/\partial n \, , \quad M_n \, , \quad V_n \tag{9}$$

Here n represents the external normal to the boundary curve and M_n is the bending moment and V_n is the transverse shear resultant which is represented by the Kirchhoff condition in the classical plate theory.

In what follows, first a general solution of equation (4) is found which satisfies initial conditions (8) and some arbitrary boundary conditions.

Dirac-delta function $\delta(x - x_o)$ is defined by:

$$\delta(x - x_o) = \begin{cases} \infty & \text{for } x = x_o \\ 0 & \text{for } x \neq x_o \end{cases} \quad \text{and} \int_{(A)} \delta(x-x_o)\delta(y-y_o)\,dA = 1 \tag{10}$$

where the double integral in equation (10) is taken over the whole area A of the plate. After Schwartz's book [7], the old argument about the existence of such a function can be looked upon as a solved problem.

Let the time dependent concentrated load $F(t)$ be applied to the point with coordinate (x_o, y_o). Differential equation (4) can be written in a convenient form by using equations (6) and (10):

$$D\Delta^2 w + (\rho/g)\,\ddot{w} = \delta(x-x_o)\,\delta(y-y_o)\,F(t) \tag{11}$$

The solution of this equation is a linear, homogeneous functional of $F(t)$. Thus:

$$w(x,y,t) = \int_o^t G(x,y,t-\tau)\,F(\tau)\,d\tau \tag{12}$$

Substitution of equation (12) into equations (11) and (8) gives:

$$D\Delta^2 G + (\rho/g)\,\ddot{G} = 0 \tag{13}$$

$$G(x,y,0) = 0\,, \quad (\rho/g)\dot{G}(x,y,0) = \delta(x-x_o)\,\delta(y-y_o) \tag{14}$$

The general solution of equation (13) satisfying the first initial condition given by the first of equations (14) and all boundary conditions is:

$$G = \sum_{m,n} a_{mn}\, g_{mn}\,(x,y) \sin \alpha_{mn} t \quad (m,n = 1,2,\ldots) \tag{15}$$

where a_{mn} are arbitrary, and $g_{mn}(x,y)$ are free of arbitrary constants and satisfy all boundary conditions and the differential equation:

$$\Delta^2 g_{mn} - p_{mn}{}^4 g_{mn} = 0 , \qquad (16) \qquad p_{mn}{}^4 = \rho\alpha_{mn}{}^2/gD \qquad (17)$$

Functions g_{mn} are the eigenfunctions. Quantities p_{mn} are the eigenvalues and are determined from a frequency equation which will be known when the boundary conditions are explicitly given. a_{mn} must now be determined from the second of equations (14):

$$\sum_{m,n} (\rho/g) a_{mn}\alpha_{mn}g_{mn}(x,y) = \delta(x-x_o)\,\delta(y-y_o) \qquad (18)$$

When the eigenfunctions are not orthogonal this represents a difficult problem and there is no universal method of determining the a_{mn} . In many cases it is possible to use the method used for orthogonal eigenfunctions except that the series (15) must be employed to sum the resulting series. In many important practical problems the eigenfunctions are orthogonal. Consequently, a_{mn} can be determined simply by multiplying equation (18) by g_{rs} , integrating over the total area of the plate and considering the condition of orthogonality:

$$\int_{(A)} g_{mn} g_{rs}\, dA = \begin{cases} b_{mn} & \text{for } r = m, \quad n = s \\ 0 & \text{for } r \neq m \neq n \neq s \end{cases} \tag{19}$$

Hence:

$$a_{mn} = g_{mn}(x_o, y_o)\; g/\rho\; b_{mn} \alpha_{mn} \tag{20}$$

Consequently:

$$w(x,y,t) = (g/\rho) \sum_{m,n} (1/b_{mn}) g_{mn}(x_o, y_o)\; g_{mn}(x,y)$$

$$\int_o^t (1/\alpha_{mn})\; F(\tau) \sin \alpha_{mn}(t-\tau)\, d\tau \tag{21}$$

The deflection at the point of contact is therefore found to be:

$$w_o = (2/M) \sum_{m,n} c_{mn} \int_o^t (1/\alpha_{mn})\; F(\tau)\; \sin \alpha_{mn}(t-\tau)\; d\tau \tag{22}$$

$$c_{mn} = g_{mn}{}^2(x_o, y_o)\; A/2b_{mn} \tag{23}$$

where A is the total area in case of plates and total length in case of beams, and M is the total mass.

When the free vibration problems of beams or plates are solved the eigenfunctions g_{mn} and eigenvalues p_{mn} (consequently α_{mn}) will be known. Thus, the determination of b_{mn} from equation (19) and c_{mn} from (23) presents no difficulty.

A few practical examples for beams and plates are given below. In all examples given below, the eigenfunctions are represented by g_n or g_{mn} and the frequency equation is marked by (F.E.). Symbols n and m represent positive integers.

a) Beams. Eigenfunctions and eigenvalues of the following examples are known [8]. Thus, b_n and c_n are calculated by using Equations (19) and (23) above, since in all cases the eigenfunctions are orthogonal. The origin is at one end of the beam.

1) Simply supported beams

$$g_n(x) = \sin n\pi x/L \ , \ \alpha_n = n^4\pi^4 g\ EI/\rho\ L^4 \quad \text{(F.E.)} \tag{24}$$

$$b_n = L/2 \quad , \quad c_n = \sin^2 n\pi x_o/L$$

2) Beams having both ends clamped

$$\left.\begin{aligned} g_n(x) &= \frac{\sinh\frac{1}{2}p_n(2x-L)}{\sinh\frac{1}{2}p_nL} - \frac{\sin\frac{1}{2}p_n(2x-L)}{\sin\frac{1}{2}p_nL} \ , \\ \cos p_nL &= \operatorname{sech} p_nL \qquad \text{(F.E.)} \\ b_n &= \frac{1}{p_n}\left[\frac{p_nL + \sinh p_nL}{1-\cosh p_nL} + \frac{p_nL + \sin p_nL}{1-\cos p_nL}\right] \ , \\ c_n &= g_n^{\ 2}(x_o)L/2\ b_n \end{aligned}\right\} \tag{25}$$

3) Cantilever beams clamped at $x = 0$

$$\left.\begin{aligned}
g_n(x) &= \frac{\cosh p_n x - \cos p_n x}{\cosh p_n L + \cos p_n L} - \frac{\sinh p_n x - \sin p_n x}{\sinh p_n L + \sin p_n L}, \\
\cos p_n L &= -\operatorname{sech} p_n L \qquad \text{(F.E.)} \\
b_n &= p_n^{-1} (\cosh p_n L - \cos p_n L)^{-2} \left(p_n L - \frac{\cos p_n L - \cosh p_n L}{\sin p_n L + \sinh p_n L}\right) \\
c_n &= g_n^{2} (x_o)\, L/2b_n
\end{aligned}\right\} \quad (26)$$

4) Beams pinned at $x = 0$ and clamped at $x = L$

$$\left.\begin{aligned}
g_n(x) &= \frac{\sinh p_n x}{\sinh p_n L} - \frac{\sin p_n x}{\sin p_n L}, \\
\tan p_n L &= \tanh p_n L \qquad \text{(F.E.)} \\
b_n &= \frac{L}{2} (\sin^{-2} p_n L - \sinh^{-2} p_n L), \\
c_n &= g_n^{2}(x_o)\, L/2b_n
\end{aligned}\right\} \quad (27)$$

b) Circular Plates. Eigenfunctions and eigenvalues of the following examples are known [8]. Equations (19) and (23) above give b_n and c_n, respectively, since in both of these cases the eigenfunctions are orthogonal. The origin is taken at the center of the plate. The outer radius is a. Polar coordinates r and θ are used.

1) Simply supported circular plate

$$g_{mn}(r,\theta) = [J_n(p_m r) + B\, I_n(p_m r)] \sin\, (n\theta + \gamma_o) \quad ,$$

$$B = -\, J_n(p_m a)/I_n(p_m a) \tag{28}$$

where J_n and I_n are the Bessel functions of the first kind with real and imaginary arguments.

$$\frac{I_{n+1}\,(p_m a)}{I_n\,(p_m a)} + \frac{J_{n+1}\,(p_m a)}{J_n\,(p_m a)} = \frac{2p_m a}{1-\nu} \quad \text{(F.E.)} \tag{29}$$

Here ν is the Poisson's ratio.

$$b_{mn} = 2\pi a^2\, J_n{}^2(p_n a)\, [\frac{1 + \nu + 2n}{\nu - 1} - 2(\frac{p_m a}{1-\nu}) + 2\frac{p_m a}{1-\nu}\, \frac{J_{n+1}(p_m a)}{J_n\,(p_m a)}\,]$$

$$c_{mn} = g_{mn}{}^2\, (r_o,\, \theta_o)\, \pi a^2/2b_{mn} \tag{30}$$

2) Circular plate clamped at the outer edge

$$\left.\begin{aligned} g_{mn}(r,\theta) &= [J_n(p_m r) + B\, I_n(p_m r)] \sin\, (n\theta + \gamma_o) \quad , \\ B &= -\, J_n(p_m a)/I_n(p_m a) \end{aligned}\right\} \tag{31}$$

$$\frac{J_{n+1}\,(p_m a)}{J_n\,(p_m a)} + \frac{I_{n+1}\,(p_m a)}{I_n\,(p_m a)} = 0 \quad \text{(F.E.)} \tag{32}$$

$$b_{mn} = 2\pi a^2 J_n{}^2(p_m a) \quad , \quad c_{mn} = g_{mn}{}^2(r_o,\theta_o)/4J_n{}^2(p_m a)$$

(33)

c) Rectangular Plates. For rectangular plates a quick method similar to the one used for beams can be used to obtain boundary conditions which will produce orthogonal eigenfunctions.

The following is constructed from Equation (16):

$$\int_o^b \int_o^a (g_{rs}\Delta^2 g_{mn} - g_{mn}\Delta^2 g_{rs})\,dxdy = (p_{mn}{}^4 - p_{rs}{}^4)\int_o^b\int_o^a g_{mn}g_{rs}\,dxdy \tag{34}$$

Integration by parts of the left side of this equation leads to:

$$(p_{mn}{}^4 - p_{rs}{}^4)\int_o^b\int_o^a g_{mn}g_{rs}\,dxdy = \int_o^b \Big[g_{rs}\frac{\partial^3 g_{mn}}{\partial x^3} - \frac{\partial g_{rs}}{\partial x}\frac{\partial^2 g_{mn}}{\partial x^2} \tag{35}$$

$$+ \frac{\partial^2 g_{rs}}{\partial x^2}\frac{\partial g_{mn}}{\partial x} - \frac{\partial^3 g_{rs}}{\partial x^3} g_{mn} + \frac{\partial^2 g_{rs}}{\partial y^2}\frac{\partial g_{mn}}{\partial x} \quad \frac{\partial^3 g_{rs}}{\partial y^2 \partial x}\Big]_{x=0}^{x=a} dy$$

and a similar equation for $y = 0$ and b, which may be obtained by interchanging x with y. The bracket on the right side of equation (35) is zero for two types of practical boundaries -- namely, simple support and clamped edge. Consequently, these two types of support conditions in any order lead to normal modes. Many other conditions can be obtained by making the bracket on the right side of Equation (35) zero, with the proper choice of g and its derivatives. The following two examples are most practical:

1) <u>Rectangular plates simply supported at all edges.</u> Two perpendicular sides with lengths a and b are taken as the coordinate axes, origin being at a corner:

$$\left.\begin{aligned}
&g_{mn}(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad \alpha_{mn}{}^{2} = \frac{gD}{\rho}\,p_{mn}{}^{4} \qquad \text{(F.E.)}\\
&p_{mn}{}^{2} = [(m\pi/a)^{2} + (n\pi/b)^{2}], \quad b_{mn} = ab/4,\\
&c_{mn} = 2\left(\sin\frac{m\pi x_{o}}{a}\sin\frac{n\pi y_{o}}{b}\right)^{2}
\end{aligned}\right\} \quad (36)$$

2) <u>Rectangular plate simply supported along $x = 0, a$; clamped along $y = 0, b$.</u>

$$\left.\begin{aligned}
&g_{mn}(x,y) = \cosh\alpha y - \cos\beta y - B\left(\sin\beta y - \frac{\beta}{\alpha}\sinh\alpha y\right)\\
&B = (\cosh\alpha b - \cos\beta b)/\left(\cos\beta b - \frac{\beta}{\alpha}\sinh\alpha b\right)\\
&\alpha = [p^{2} + (m\pi/a)^{2}]^{\frac{1}{2}}, \quad \beta = [p^{2} - (m\pi/a)^{2}]^{\frac{1}{2}}
\end{aligned}\right\} \quad (37)$$

$$(\cos\beta b - \cosh\alpha b)^{2} + \left(\sin\beta b - \frac{\beta}{\alpha}\sinh\alpha b\right)\left(\sin\beta b + \frac{\alpha}{\beta}\sinh\alpha b\right) = 0 \qquad \text{(F.E.)} \qquad (38)$$

$$
\left.\begin{aligned}
b_{mn} &= \tfrac{1}{4}ab\left(1 + \frac{\beta^2}{\alpha^2}\right) + \tfrac{1}{4}\frac{a}{\alpha}\left(\sin\beta b - \frac{\beta}{\alpha}\sinh\alpha b\right)^{-1} \\
&\quad \cdot \left\{\left[\left(1 - \frac{\beta^2}{\alpha^2}\right)\sinh\alpha b\sin\beta b - 2\frac{\beta}{\alpha}\cosh\alpha b\cos\beta b\right]\right. \\
&\quad \left.\left(\cosh\alpha b + \frac{\alpha^2}{\beta^2}\cos\alpha b\right) + 2\frac{\alpha}{\beta}\left(\cosh\alpha b + \frac{\beta^2}{\alpha^2}\cos\beta b\right)\right\}, \\
c_{mn} &= g_{mn}^{\;2}(x_o, y_o)\, ab/2b_{mn}
\end{aligned}\right\} \qquad (39)
$$

4. The Integral Equation

In the previous chapter it is shown that in all impact problems the flexural deflection at the point of contact is given by Equation (22). The integral equation (7) of impact can be written in a common non-dimensional form as:

$$L(f) \equiv af^{2/3}(x) - x + \int_0^x K(x-y)\, f(y)\, dy = 0 \tag{40}$$

$$\left.\begin{aligned} &f(x) = T_o F(T_o x)/mv_o \quad , \quad K(x) = x + 2\mu \sum_{m,n} \frac{c_{mn}}{\omega_{mn}} \sin\omega_{mn} x \ , \\ &\mu = m/M \quad , \quad x = t/T_o \quad , \quad a = k\, m^{2/3} v_o^{-1/3}\, T_o^{-5/3} \\ &\omega_{mn} = \alpha_{mn} T_o \end{aligned}\right\} \tag{41}$$

where T_o may be chosen in a suitable manner. There c_{mn} and ω_{mn} are given by the solution of the flexure problem.

Another form of Equation (40), which will be used later, is:

$$\begin{aligned} &a\phi(x) - x + \int_0^x K(x-y)\, \phi^{3/2}(y)\, dy = 0 \ , \\ &\phi(x) = f^{2/3}(x) \end{aligned} \tag{42}$$

5. Approximate Solutions by Generalized Galerkin Method

The Generalized Galerkin method is developed in [1]. This method gives an approximate solution for any problem having an equation:

$$L(f) = 0 \tag{43}$$

Let a function f which depends on the arbitrary independent infinitely many parameters $(a_1, a_2, \ldots)$ and independent variable x, be chosen such that either $\partial f/\partial a_i$ satisfy the boundary conditions at $x = x_1$ and $x = x_2$, imposed on equation (43) or $L(f) = 0$ there. This function $f(x, a_1, a_2, \ldots)$ obviously will not satisfy equation (43). This function represents a solution of equation (43) if a_i are solutions of the following set of non-linear algebraic equations:

$$\int_{x_1}^{x_2} L[f(x, a_1, a_2, \ldots)] \frac{\partial f}{\partial a_i} dx = 0 \quad (i = 1, 2, \ldots) \tag{44}$$

When i is finite $f(x, a_1, a_2, \ldots, a_N)$ is an approximate solution.

Various functions of the following type are suitable to use as $f(x, a_1, a_2, \ldots)$:

(a) $f = a_1 \sin a_2 x$

(b) $f = a_1 \sin^2 a_2 x$

(c) $f = a_1 \sin^{3/2} a_2 x$

(d) $f = \sum_{i=1}^{N-1} a_i \sin i\, a_N x$

Hence equation (44) takes the form:

$$\int_0^{\pi/a_N} L(f)\ (\partial f/\partial a_i)dx = 0 \quad (i = 1,2,\ldots,N) \tag{45}$$

Note that $\partial f/\partial a_1$ and $\partial f/\partial a_2$ for cases (b) and (c) satisfy the end conditions at $x = 0$, π/a_2 . However, in case (a), one end condition is violated since $\partial f/\partial a_2$ does not vanish at $x = \pi/a_2$, end of the contact. Nevertheless, in all examples worked out, very good approximations are obtained. Cases (b) and (c) give better results than case (a). (See Figures 18, 19, and 20.) In case (d) a_N is chosen the same as a_2 of case (a) since a_2 of case (a) is found to be nearly equal to its exact value.

Explicit forms of Equation (45) are given below for cases (a), (b), and (d). Computations are carried out for cases (a), (c), and (d).

<u>Case (a)</u>

$$\left.\begin{aligned} &(\tfrac{1}{2} + 2\mu\alpha_1)\ \bar{A}_1 + d.\gamma\cdot\bar{A}_1^{2/3} = 1 \\ &(\tfrac{7}{8} - \tfrac{1}{4}\mu\,\delta)\ \bar{A}_1 + \tfrac{3}{10}\,d\cdot\gamma\bar{A}_1^{2/3} = 1 \end{aligned}\right\} \tag{46}$$

Elimination of $\bar{A}_1^{2/3}$ between these two equations leads to:

$$\left.\begin{aligned} \bar{A}_1 &= 28/[29 - \mu\ (10\delta + 24\alpha_1)] \\ \gamma = a\ a_2^{5/3}/\pi &= \frac{15 - 10\mu\ (\delta + 8\alpha_1)}{d\ 28^{2/3}[29 - \mu\ (10\delta + 24\alpha_1)]^{1/3}} \end{aligned}\right\} \quad (47)$$

Case (b)

$$\left.\begin{aligned} &(\tfrac{\pi}{6} - \tfrac{5}{8\pi} + \tfrac{\mu}{4\pi}\alpha_1{}') \bar{A}_1 + d'\gamma\ \bar{A}_1^{2/3} = 1 \\ &(\tfrac{\pi}{4} - \tfrac{9}{32\pi} + \tfrac{\mu}{4\pi}\delta') \bar{A}_1 + \tfrac{3}{10} d'\gamma\ \bar{A}_1^{2/3} = 1 \end{aligned}\right\} \quad (48)$$

Similarly, $\bar{A}_1^{2/3}$ can be eliminated from these equations to obtain equations resembling equations (47).

Case (d)

$$\sum_{i, i \neq j} (1 + 2\mu\beta_{ij})\ \bar{A}_i + [1 - \tfrac{1}{2}(-1)^{j+1}]\ \bar{A}_j + 2\mu\alpha_j \cdot \bar{A}_j$$

$$+ (-1)^{j+1}\ \eta_j\ \bar{A}_1^{2/3} = 1 \quad (j = 1,2,\ldots) \qquad (49)$$

where:

$$\alpha_i = \beta_{ii} + (-1)^i \tfrac{1}{2} \beta_i = \sum_{m,n} c_{mn} A(\lambda_{mn}),$$

$$\delta = \beta_1 - 4\delta_1 = \sum_{m,n} c_{mn} D(\lambda_{mn})$$

$$\gamma = a\, a_2^{5/3}/\pi \quad , \quad d = \pi^{1/2}\, \Gamma(4/3)/\Gamma(11/3) \cong 1.68257$$

$$\delta_1 = \sum_{m,n} c_{mn} D_1(\lambda_{mn}) \quad , \quad \beta_i = \sum_{m,n} c_{mn} B_i(\lambda_{mn}) \quad ,$$

$$\beta_{ij} = \sum_{m,n} c_{mn} B_{ij}(\lambda_{mn})$$

$$D_1(\lambda_{mn}) = (1 - \lambda_{mn}^2)^{-2} \cos\pi\lambda_{mn} + (1 - \lambda_{mn}^2)^{-3}$$

$$\cdot\, (1 + \lambda_{mn}^2)(\pi\lambda_{mn})^{-1} \sin\pi\lambda_{mn} \tag{50}$$

$$B_i(\lambda_{mn}) = [1 - (\lambda_{mn}/i)^2]^{-1}$$

$$B_{ij}(\lambda_{mn}) = [(1 - (\lambda_{mn}/i)^2]^{-1} [1 - (\lambda_{mn}/j)^2]^{-1}$$

$$(\pi\lambda_{mn})^{-1} \sin\pi\lambda_{mn}$$

$$\overline{A}_i = a_i/a_N \quad , \quad \lambda_{mn} = \omega_{mn}/a_N \quad (i,j = 1,2,\ldots)$$

$$\left.\begin{aligned}
&\alpha_1' = \sum_{m,n} c_{mn} A'(\lambda_{mn}) \quad , \quad \delta' = \sum_{m,n} c_{mn} D'(\lambda_{mn}) \\
&A'(\lambda_{mn}) = 2(2/\lambda_{mn})^2 - (1 - \tfrac{1}{4}\lambda_{mn}{}^2)^{-1} - \tfrac{1}{\pi}(\tfrac{1}{2}\lambda_{mn})^{-3} \\
&\qquad \cdot (1 - \tfrac{1}{4}\lambda_{mn}{}^2)^{-2} \sin\pi\lambda_{mn} \\
&D'(\lambda_{mn}) = (2/\lambda_{mn})^2 + \tfrac{1}{4}(1 - \tfrac{1}{4}\lambda_{mn}{}^2)^{-1} - \tfrac{1}{\pi}(2/\lambda_{mn})(1 - \tfrac{1}{4}\lambda_{mn}{}^2)^{-3} \\
&\qquad \cdot \sin\pi\lambda_{mn} - (2/\lambda_{mn})^2(1 - \tfrac{1}{4}\lambda_{mn}{}^2)^{-2}\cos\pi\lambda_{mn} \\
&d' = (14/5\pi^{1/2})\,\Gamma(7/6)\,/\,\Gamma(5/3)
\end{aligned}\right\} \quad (51)$$

$$\bar{A}_1{}^{2/3}\eta_j = \int_0^\pi (\sum_i \bar{A}_i \sin iu)^{2/3} \sin ju\, du \quad (i,j = 1,2,\ldots,N) \qquad (52)$$

For N = 3, η_i have the explicit forms given below which are obtained by using the binomial expansion.

$$\left.\begin{aligned}
\eta_1 &= 1.682621 + 0.101974\ (3\ \bar{A}_3/\bar{A}_1) \\
\eta_2 &= 1.223724\ (2\ \bar{A}_2/\bar{A}_1) \\
\eta_3 &= 0.152967 + 1.265714\ (3\ \bar{A}_3/\bar{A}_1)
\end{aligned}\right\} \quad (53)$$

$A_i(\lambda_{mn})$, $D(\lambda_{mn})$, $B_i(\lambda_{mn})$, $B_{ij}(\lambda_{mn})$, $(i \neq j)$, $A'(\lambda_{mn})$, and $D'(\lambda_{mn})$ are plotted on Figures 2 to 5. Hence for any impact problem these curves can be used to sum up the series involved in the expressions of α_i , β_{ij} and δ . If the contact time π/a_N is chosen by an approximate method as suggested by Zener and Feshbach and E. H. Lee [1] or simply taken as the Hertz time (as this seems to give a better approximation) then these series can be computed. One can then, for example in case (a), use the first of equation (46) to determine the maximum contact force $\bar{A}_1$. This, of course, assumes that the second equation of (46) is satisfied. Since this will not be the case in practice, this reference contact time π/a_2 may be used on the right sides of equation (46) to obtain a better a_2 and $\bar{A}_1$.

For particular beams and plates this process may be further shortened by giving curves for δ_1 , α_i , β_{ij} . In this report this is done for the following simply supported plate problems under central impact:

(i) Circular plate, Fig. 6

(ii) Square plate, Fig. 7

(iii) Rectangular plate, with a/b = 2, Fig. 7

(iv) Rectangular plate with a/b = 4, Fig. 7

Similar curves were given in the previous report for simply supported beams under central impact (Fig. 20, [1]). For other types of beam and plate problems this may be done once and for all, reducing the problem of impact to a routine one. Further discussion and application of this method to various problems are given in Chapter 8.

6. Collocation Method

This method consists of satisfying the integral equation (40) or (42) at various characteristic points. In a contact force curve the important points are: origin, the point of maximum, the end of the contact time. The contact force curve differs slightly from a symmetric shape. Thus, choice of a symmetric curve is reasonable. Let x_c be the unknown non-dimensional contact time. Also let

$$f(x) = A^3 X(x) \quad , \quad X(0) = X(x_c) = 0 \quad , \quad X(x_c/2) = 1$$
$$S(x) = \int_0^x K(x-y)\; X(y)\,dy \tag{54}$$

where A^3 is the unknown maximum aplitude. Equation (40) must be satisfied at $x = x_c/2$. Also df/dx obtained from Equation (40) must be zero at this point. This leads to two equations to evaluate A and x_c , namely:

$$\left.\begin{aligned} A^3 &= S_{,x}{}^{-1}(x_c/2) \\ x_c/2 &= a\, S_{,x}{}^{-2/3}\;(x_c/2) + S(x_c/2)\; S_{,x}{}^{-1}(x_c/2) \end{aligned}\right\} \tag{55}$$

where:

$$S_{,x}(x_c/2) = [\tfrac{d}{dx}\, S(x)]_{x=x_c/2}$$

Equations (55) are also valid if one uses Equation (42) and choses:

$$\phi(x) = A^2 Y(x) \quad , \quad Y(0) = Y(x_c) = 0 \quad , \quad Y(x_c/2) = 1$$

$$S(x) = \int_0^x K(x-y)\, Y^{3/2}(y)\, dy \tag{56}$$

Hence, if a shape function $x(x)$ or $Y(x)$, having the conditions stated in Equation (54) or (56), is chosen, A and x_c can be calculated from Equation (55).

Three practical shape functions are:

(a) $X(x) = \sin \pi x/x_c$, (b) $X(x) = \sin^2 \pi x/x_c$,

(c) $Y(x) = \sin \pi x/x_c$

The results for S and $S_{,x}$ in these cases may be written:

$$S(x_c/2) = (x_c/\pi)^2 (d_1 + 2\mu R_1) \; ,$$

$$S_{,x}(x_c/2) = (x_c/\pi)(d_2 + 2\mu R_2) \tag{57}$$

where:

Case (a)

$$\left.\begin{aligned}
&d_1 = \frac{\pi}{2} - 1 \quad , \quad d_2 = 1 \quad , \quad R_i = \sum_{m,n} c_{mn}\rho_i(\lambda_{mn}) \quad (i = 1,2) \\
&\rho_1(\lambda_{mn}) = (1 - \lambda_{mn}{}^2)^{-1} (\lambda_{mn}{}^{-1} \sin \tfrac{1}{2}\pi\lambda_{mn} - 1) \\
&\rho_2(\lambda_{mn}) = (1 - \lambda_{mn}{}^2)^{-1} \cos \tfrac{1}{2}\pi\lambda_{mn} \\
&\lambda_{mn} = \omega_{mn}\, x_c/\pi
\end{aligned}\right\} \quad (58)$$

Case (b)

$$\left.\begin{aligned}
&d_1 = (\pi^2 - 4)/16 \quad , \quad d_2 = \pi/4 \\
&\rho_1(\lambda_{mn}) = \tfrac{1}{8}[4\lambda_{mn}{}^{-2} - (1 - \tfrac{1}{4}\lambda_{mn}{}^2)^{-1} (1 - 4\lambda_{mn}{}^{-2} \cos \tfrac{1}{2}\pi\lambda_{mn})] \\
&\rho_2(\lambda_{mn}) = \lambda_{mn}{}^{-1}(1 - \tfrac{1}{4}\lambda_{mn}{}^2)^{-1} \sin \tfrac{1}{2}\pi\lambda_{mn}
\end{aligned}\right\} \quad (59)$$

Case (c)

$$\left.\begin{aligned}
&d_1 = 0.457906 \quad , \quad R_1 = 0.915310\, R_1{}' - 0.011300 R_1{}'' \\
&d_2 = 0.811409 \quad , \quad R_2 = 0.915310\, R_2{}' - 0.033901 R_2{}'' \\
&R_1{}' = \text{same as } R_1 \text{ of case (a)}; \; R_2{}' = \text{same as } R_2 \text{ of case (a)} \\
&R_1{}'' = \sum_{m,n} c_{mn}\rho_1{}'(\lambda_{mn}), \; R_2{}'' = \sum_{m,n} c_{mn}\rho_2{}'(\lambda_{mn}) \\
&\rho_1{}'(\lambda_{mn}) = [1 - (\lambda_{mn}/3)^2]^{-1} [(3/\lambda_{mn}) \sin \tfrac{1}{2}\pi\lambda_{mn} + 1] \\
&\rho_2{}'(\lambda_{mn}) = [1 - (\lambda_{mn}/3)^2]^{-1} \cos \tfrac{1}{2}\pi\lambda_{mn}
\end{aligned}\right\} \quad (60)$$

Quantities $\rho_1(\lambda_{mn})$, $\rho_2(\lambda_{mn})$, $\rho_1'(\lambda_{mn})$, $\rho_2'(\lambda_{mn})$ given by equations (58) to (60) are plotted as functions of λ_{mn} in Figures 8 to 12. Consequently, for any type of plate problems, series R_1 and R_2 can be obtained by summing the individual terms given by these figures. In particular, these series summed for simply-supported circular plates and simply-supported rectangular plates with the side ratios a/b = 1, 2 and 4. Consequently, for the latter plates under central impact we can read the values of R_1, R_2, R_1' and R_2' from Figures 13 to 17, or from Tables 4, 5 and 6. Hence, in computation, we first select a contact time x_c then compute λ_{10} from the last of equation (58). Next we read R_1 and R_2 from these figures. Using equation (57) we calculate the right side of the second of equations (55). If this comes out to be $x_c/2$ which we chose to start with, the calculation is finished. A^3 is then calculated by the first of equation (55). If it does not we try another $x_c/2$ value. Usually, two or three trials are sufficient to determine x_c satisfying the second of Equations (55). This method is faster than the generalized Galerkin method. It is not, however, as general and consequently, in some extreme cases of impact problems, such as multiple impact, it may not be as satisfactory as the other method. Computations are made for circular and rectangular plates using forcing function shapes given by cases (a) and (c). The results are plotted on figures 18 to 20 next to those obtained by the Generalized Galerkin method.

In the case of square plates, the result of K. Karas [6] obtained by the reliable stepwise integration method is plotted in Fig. 19. An examination of these curves indicate that the approximation offered to the impact problems is satisfactory for all engineering purposes. Further example calculations are given in Chapter 8.

7. Deflection and Flexural Stresses

Deflection w can be calculated from equation (21) as soon as F(t) is determined by one of the previous methods.

All equations obtained for contact force can be put into the form:

$$F(t) = \sum_{i=1,2,\ldots} F_i \sin\lambda_i t \qquad 0 \leq t \leq T_c$$
$$F(t) = 0 \qquad t \geq T_c \tag{61}$$

where F_i and λ_i are known or can be determined from the expressions of the contact force.

Substitution of the Equation (61) into equation (21) leads to:

$$w(x,y,t) = (2/M)\sum_{m,n,i} c_{mn} [g(x,y)/g(x_o,y_o)] (F_i/\lambda_i^2)$$

$$[1 - (\alpha_{mn}/\lambda_i)^2]^{-1} [(\lambda_i/\alpha_{mn}) \sin\alpha_{mn}t - \sin\lambda_i t],$$

$$\text{For } 0 \leq t \leq T_c \tag{62}$$

$$w(x,y,t) = (2/M)\sum_{m,n} c_{mn}[g(x,y)/g(x_o,y_o)][B_{mn}\sin\alpha_{mn}t$$

$$+ C_{mn} \cos\alpha_{mn}t], \quad \text{For } t \geq T_c \tag{63}$$

where:

$$\left.\begin{aligned} B_{mn} &= \sum_i (F_i/\lambda_i^2) [1 - (\alpha_{mn}/\lambda_i)^2]^{-1} [1-(-1)^i \cos\alpha_{mn}T_c](\lambda_i/\alpha_{mn}) \\ C_{mn} &= \sum_i (F_i/\lambda_i^2)[1 - (\alpha_{mn}/\lambda_i)^2]^{-1} (-1)^i(\lambda_i/\alpha_{mn}) \sin\alpha_{mn}T_c \end{aligned}\right\} \tag{64}$$

The deflection at the point of contact is obtained by taking $x = x_o$, $y = y_o$, in Equations (62) and (63).

The flexural stresses σ_x, σ_y and shear stresses τ_{xz} and τ_{yz} can be determined from:

$$\left.\begin{aligned} \sigma_x &= 6M_x/h^2 \quad , \; \tau_{xz} = 3Q_x/2h \\ \sigma_y &= 6M_y/h^2 \quad , \; \tau_{yz} = 3Q_y/2h \end{aligned}\right\} \quad (65)$$

where:

$$\left.\begin{aligned} M_y &= -D(w_{,xx} + \nu w_{,yy}), \quad M_y = -D(w_{,yy} + \nu w_{,xx}) \\ M_{xy} &= -M_{yx} = -D(1-\nu)w_{,xy} \\ Q_x &= -D(w_{,xxx} + w_{,yyx}) \; , \quad Q_y = -D\,(w_{,xxy} + w_{,yyx}) \end{aligned}\right\} \quad (66)$$

Here M_x, M_y are bending moments, M_{xy} twisting moment, Q_x and Q_y are the transverse shear forces, all per unit length. The flexural stresses contain second and third derivatives of deflection, consequently, convergence is very slow. As a matter of fact, τ_{xy} and τ_{yz} diverge at the point of contact. This fact is well known in plate theory. Bending stresses σ_x and σ_y are slowly convergent everywhere. This is, of course, the case even for the static deflection under a concentrated load.

As pointed out in the previous report, [1], the shape of the contact force has little influence on the deflection of the plate. As a matter of fact, if one considers the extreme case of the Dirac δ-function type contact force having the same impulse as F(t) and place it at the half of the contact time interval, one finds that the deflection produced this way differs little from the actual deflection, Fig. 22. Of course, a shape closer to the actual shape will give better results. Therefore, the maximum error in deflection would be reached by considering a δ-function type of contact force. This fact also indicates that the deflections cannot be used as a criterion of accuracy for the shape of F (t). The situation for stresses is, of course, different.

Deflection at the centers of a simply-supported square plate and of a circular plate due to a central impact are computed. The result is plotted in Figures 21 and 22. The result for a δ-function type contact force for the case of the square plate is also plotted in Fig. 22. Deflection given by K. Karas [6], based on the reliable step-by-step integration method which is also drawn in Fig. 22, compares very well with these approximate solutions.

8. Illustrative Examples

Calculations for the case of central impact on simply-supported beams are given in our previous report. In the present report, examples are worked out for the case of central impact on

(i) Simply-supported circular plates

(ii) Simply-supported rectangular plates

having size ratios $a/b = 1$, 2 and 4.

Both the Generalized Galerkin method as well as the Collocation method are used to determine the contact force and contact time. In all problems the striking body is assumed to be a sphere. This, of course, is immaterial as far as the theory is concerned, since all we need is to require that the striking body should have a known radius of curvature at the point of contact, in accordance with the Hertz theory of impact. Once this radius of curvature is known, the Hertz coefficient k can be determined [4]. For a sphere striking a plate the equation for k is particularly simple and is given by:

$$k = 1.230520 \, (E^2 r_s)^{-1/3} \tag{67}$$

where E is the common moduli of elasticity of sphere and plate r_s is the radius of the sphere. In all the calculations, the following quantities are taken the same:

$$E = 2.2 \times 10^6 \text{ kg/cm}^2 \quad , \quad \nu = 0.3$$

$$r_s = 1.00 \text{ cm} \quad , \quad h = 0.8 \text{ cm}$$

$$v_o = 100 \text{ cm/sec} \quad , \quad \gamma = 0.00796 \text{ kg/cm}^3$$

where ν is the common Poisson's ratio of both plates and spheres, h is the plate thickness and γ is the common weight density of the plate and the sphere. Hence k, for the examples of this report, is given by:

$$k = 0.727456 \times 10^{-4}\ \mathrm{cm/kg}^{2/3}$$

(i) Circular Plate simply-supported at the outer edge:

Radius of the plate = a = 10cm.

Values of $p_m a$ which are the roots of the frequency equation (29) are tabulated in Table 1. α_m and c_m are calculated from equations (17) and (30) where for the case of central impact in expression (28) of $g_{mn}(r,\theta)$ we must take $n = 0$ and $\gamma_o = \pi/2$. Hence for this case we have:

$$\left.\begin{aligned} c_m &= \frac{1}{4}\left[\frac{1}{J_o(p_m a)} - \frac{1}{I_o(p_m a)}\right]\left[-\frac{1+\nu}{1-\nu} - 2\left(\frac{p_m a}{1-\nu}\right)^2 \right. \\ &\qquad \left. + 2\frac{p_m a}{1-\nu}\,\frac{J_1(p_m a)}{J_o(p_m a)}\right]^{-1} \\ \alpha_m &= (gD/\rho)^{1/2}\, p_m \end{aligned}\right\} \quad (68)$$

Values of c_m are given in Table 1 for $\nu = 0.3$.

In what follows the contact force and the contact time are given in units of kilograms and seconds.

(a) Typical Calculation for Generalized Galerkin Method

We use shape (a), that is, we assume that $f(x) = a_1 \sin a_2 x$

1st approximation: Take $\mu = 0$, then equation (47) gives:

$$\bar{A}_1 = 0.965517 \quad , \gamma = 0.314688$$

2nd approximation:

$$T_o^{-1} a_2 = (\pi\,\gamma/a)^{3/5} = 4.64944 \times 10^4, \text{ hence}$$

$$\lambda_{10} = \omega_{10}/a_2 = 0.066897 \longrightarrow \quad \text{(Fig. 6)} \quad \longrightarrow$$

$\alpha_1 = 5.35$, $\delta = -46.8$

Hence, equation (47) gives:

$\bar{A}_1 = 0.837186$, $T_o^{-1}a_2 = 4.6127 \times 10^4$, $\lambda_{1C} = 0.0674$

$\gamma = 0.310553$

Continue in this way until the difference between two subsequent values of γ or $\bar{A}_1$ or both are small enough. Even above, the second approximation gives a very satisfactory result. The values of the contact force and time are given by:

$F(t) = F_1 \sin \lambda t$, $F_1 = 131.509$ kg

$\lambda = 4.6127 \times 10^4 \text{sec}^{-1}$, $T_c = 0.681 \times 10^{-4}$ sec.

where T_c is the contact time. The result is plotted on Fig. 18.

(b) Generalized Galerkin Method having shape (d)

$$F(t) = \sum_{j=1,2,3} F_j \sin\lambda t, \text{ shape (d).}$$

Three equations given by equation (49) when solved give:

$F_1 = 133,427$ kg , $F_2 = 1.506$ kg , $F_3 = -11.981$ kg

$F_{max} = 145.408$ kg , $T_c = 0.679 \times 10^{-4}$ sec

In this calculation, λ of the foregoing calculation is used. The result is plotted in Fig. 18.

(c) Typical Calculation for the Collocation Method

We use shape (c), that is:

$$F(t) = A^3 \sin^{3/2} \pi t/T_c$$

Equation (55), upon substitution of equation (56), gives:

$$\left.\begin{aligned} A^3 &= \pi m v_o / T_c \; (d_2 + 2\mu R_2) \\ T_c^{\,5} &= \frac{k^3 m^2 v_o^{-1} \pi^2 \; (d_2 + 2\mu R_2)}{[\frac{1}{2} d_2 - \frac{1}{\pi}\, d_1 + \mu (R_2 - \frac{2}{\pi} R_1)]^3} \end{aligned}\right\} \quad (69)$$

1st approximation: Take $\mu = 0$, then Equation (69) gives:

$A^3 = 176.863$ kg , $T_c = 0.684964 \times 10^{-4}$ sec.

2nd approximation: Calculate

$\lambda_{10} = \alpha_{10} T_c / \pi = 0.067815 \longrightarrow$ Fig. 13 $\longrightarrow R_1 = 6.86$,

$R_2 = 8.42$

Substitute in equation (69) to obtain:

$A^3 = 146.457$ kg , $T_c = 0.648566 \times 10^{-4}$ sec.

Continue this way until the difference between two subsequent A^3 or/and two subsequent T_c is sufficiently small to be negligible. In the present calculation the fourth step gives the following values:

$A^3 = 146.763$ kg

$T_c = 0.645 \times 10^{-4}$ sec.

The result is plotted in Fig. 18. It compares very well with the result based on the calculation (c) above.

(ii) Rectangular Plates simply-supported at the outer edges:

The length, a, of all plates in x-direction is taken constant, a = 20 cm. The calculations are then made for:

$a/b = 1$, $a/b = 2$ and $a/b = 4$.

where b is the length of the other edge. Circular frequency

α_{mn} are given by equation (36) that is:

$$\alpha_{mn} = \alpha_{10}(m^2 + n^2\ a^2/b^2)\ ,\quad \alpha_{10} = (gD/\rho)^{1/2}\ \pi^2/a^2$$

For the present problem, $\alpha_{10} = 0.622068 \times 10^4$sec. Calculations are carried out using both methods with the results given below:

Square plate a/b = 1

$$\left.\begin{array}{ll} F = F_1 \sin \lambda\, t & \text{(shape a)} \\ F_1 = 131.782 \text{ kg} & \\ \lambda = 4.636 \times 10^4 \text{ 1/sec} & \\ T_c = 0.678 \times 10^{-4} \text{sec} & \end{array}\right\} \quad \text{(Generalized Galerkin)}$$

$$\left.\begin{array}{l} F_1 = 132.308 \text{ kg} \\ \lambda = 4.871 \times 10^4 \text{ 1/sec} \\ T_c = 0.645 \times 10^{-4} \text{ sec} \end{array}\right\} \quad \text{(Collocation (a))}$$

$$\left.\begin{array}{l} F = F_1 \sin^{3/2} \lambda t \\ F_1 = 146.763 \text{ kg} \\ \lambda = 4.875 \text{ 1/sec} \\ T_c = 0.644 \times 10^{-4} \text{sec} \end{array}\right\} \quad \text{(Collocation (c))}$$

The results are plotted in Fig. 19 next to the results of K. Karas [6], based on the reliable step-by-step integration method. The results compare very well with those of K. Karas. Shape (c) of the Collocation method can not be distinguished from the Karas result.

<u>Rectangular plate with $a/b = 2$</u>

$$\left.\begin{aligned} F &= F_1 \sin\lambda t \\ F_1 &= 130.72 \text{ kg} \\ \lambda &= 4.693 \times 10^4 \text{ 1/sec} \\ T_c &= 0.669 \times 10^{-4} \text{sec} \end{aligned}\right\} \quad \text{(Generalized Galerkin)}$$

$$\left.\begin{aligned} F_1 &= 132.680 \text{ kg} \\ \lambda &= 4.829 \times 10^4 \text{ 1/sec} \\ T_c &= 0.650 \times 10^{-4} \text{sec} \end{aligned}\right\} \quad \text{(Collocation (a))}$$

$$\left.\begin{aligned} F &= F_1 \sin^{3/2}\lambda t \\ F_1 &= 145.967 \text{ kg} \\ \lambda &= 4.844 \times 10^4 \text{ 1/sec} \\ T_c &= 0.649 \times 10^{-4} \text{sec} \end{aligned}\right\} \quad \text{(Collocation (c))}$$

See also Fig. 20.

Rectangular plate with $a/b = 4$.

Only the functions R_1 , R_2 , R_1' , R_2' , μ and δ are plotted. See Figures 7, and 14 to 17. No calculation is made for the contact force.

Circular plate clamped at the outer edge:

Table 2 gives a list of quantities $p_m a$ and c_m , the first of which is the root of frequency equation (32) and the second is given by the second of equations (33). The procedure for the calculation of the contact force is identical to those given above. No calculation has been carried out.

Deflection

The maximum deflection is computed for the circular plate and square plate as a function of time and plotted in Figs. 21 and 22. In these computations, the expressions used for the contact force are those obtained by the Generalized Galerkin method, for the circular plate and collocation (c) for the square plate; namely:

$$F(t) = 133.42735 \sin (4.627699 \times 10^4 t) + 1.505767 \sin (9.255398 \times 10^4 t) - 11.980813 \sin (13.883097 \times 10^4 t) \quad \text{(Circular Plate)}$$

$$F(t) = 146.763 \sin^{3/2} 4.875287 \times 10^4 t + 134.334 \sin (4.875287 \times 10^4 t) - 14.926 \sin (14.625861 \times 10^4 t) \text{ kg} \quad \text{(Square plate)}$$

Deflection obtained by use of a Dirac δ-function type of contact force, having the same impulse as the contact force, placed at the mid-point of the contact interval is also calculated. That is $F(t) = F_1 \, \delta(t - T_c/2)$ with $F_1 = \int_0^{T_c} F(t)dt$ is used in place of F (t). The result for the deflection is also plotted in Fig. 22.

Comparison is in favor of all the above shapes. The error committed even in the extreme case, where the δ-function is used, seems to be small.

Table 1

$p_m a$, c_m - **Simply supported circular plates,** (equation 29)

m	$p_m a$	c_m
1	2.2042	1.7397
2	5.4451	4.3309
3	8.6061	6.7909
4	11.7578	9.2557
5	14.9022	11.7270
6	18.0438	14.2004
7	21.2120	16.6556
8	24.3536	19.1249
9	27.4916	21.5844
10	30.6332	24.0505
11	33.7748	26.5164
12	36.9164	28.9909
13	40.0579	31.4606
14	43.1995	33.9271
15	46.3411	36.3924
16	49.4827	38.8631
17	52.6243	41.3186
18	55.7659	43.7099
19	58.9075	46.2670

Table 2

$p_m a$, c_m - Clamped circular plates, (equation 32)

m	$p_m a$	c_m
1	0	0
2	3.1962	2.7348
3	6.3064	4.9073
4	9.4400	7.3957
5	12.5771	9.8695
6	15.7164	12.3374
7	18.8565	14.8025
8	21.9960	17.3082
9	25.1376	19.7519
10	28.2811	22.1122
11	31.4200	24.6767
12	34.5613	27.1385
13	37.7022	29.7442
14	40.8438	32.0801
15	43.9851	34.5495
16	47.1266	37.0096
17	50.2682	39.4633
18	53.4097	41.9252
19	56.5513	44.3813
20	59.6929	46.8340

Table 3

Functions R_1, R_2 for the Collocation Method, central impact on simply supported circular plates, (equations 58-60)

λ_{10}	R_1	R_2
.040	24.1985	20.3397
.080	12.0880	10.3370
.121	7.9639	6.8764
.160	5.9109	5.1983
.200	4.8442	4.1920
.240	3.9143	3.4747
.300	3.1871	2.7443

Table 4

Functions R_1, R_2, R_1', R_2' for the Collocation Method, central impact on simply supported square plates, a/b=1, (equations 58-60)

λ_{10}	R_1	R_2	R_1'	R_2'
.08	6.4559	7.8811	7.3765	7.7704
.12	4.3901	5.2337	5.0066	5.1442
.15	---	---	---	3.9988
.16	3.3022	3.8910	3.7524	3.8175
.20	2.6825	3.2444	3.0595	3.2532
.24	2.2370	2.5896	2.5528	2.5076
.30	1.7628	1.9699	1.9959	1.8594

Table 5

Functions R_1, R_2, R_1', R_2' for the Collocation Method, central impact on simply supported rectangular plates, a/b=2, (equations 58-60)

λ_{10}	R_1	R_2	R_1'	R_2'
.06	4.3758	5.2139	5.0107	5.0937
.08	3.2067	3.7808	3.6464	3.6454
.12	2.1855	2.7429	2.4989	2.8142
.15	1.8068	2.3249	2.0810	2.4037
.18	1.5607	2.0117	1.8066	2.0628
.20	1.4267	1.7990	1.6501	1.8059
.24	1.2051	1.4247	1.3841	1.3580
.30	0.9519	1.0679	1.0770	0.9893

Table 6

Functions R_1, R_2, R_1', R_2' for the Collocation Method, central impact on simply supported rectangular plates, a/b=4, (equations 58-60)

λ_{10}	R_1	R_2	R_1'	R_2'
.06	2.3161	2.9355	2.6751	2.8738
.08	1.7519	1.9833	1.9950	1.8150
.10	1.3657	1.3978	1.5080	1.1536
.12	1.0435	0.8344	1.1369	0.5779
.14	0.8102	0.5281	0.8584	0.2954
.16	0.6319	0.2723	0.6500	0.0417
.20	0.3326	0.0213	0.3085	-0.1410
.24	0.1620	-0.0011	0.2219	-0.0590
.32	---	0.0047	---	0.0141

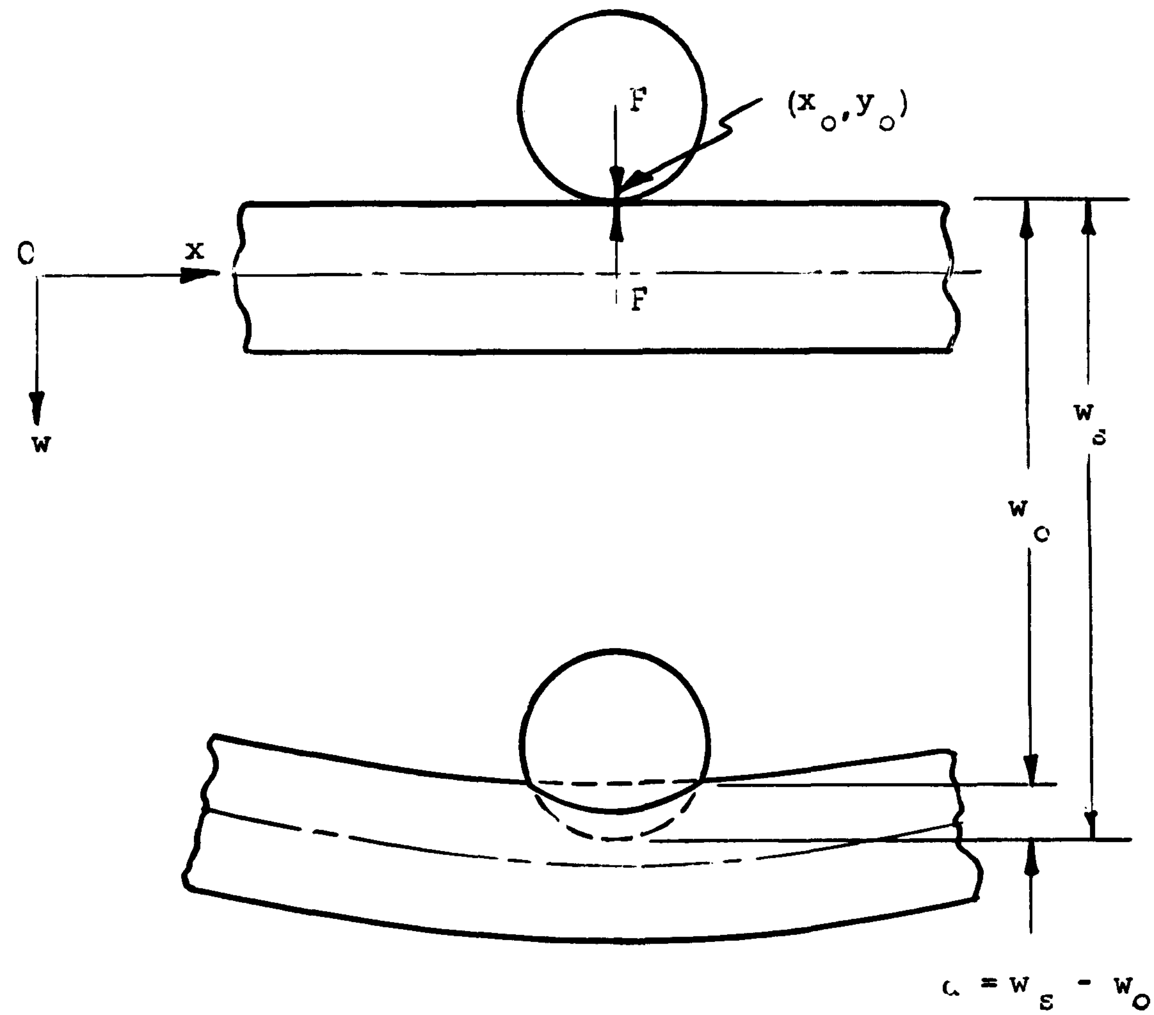

Fig. 1 Sketch of Displacements

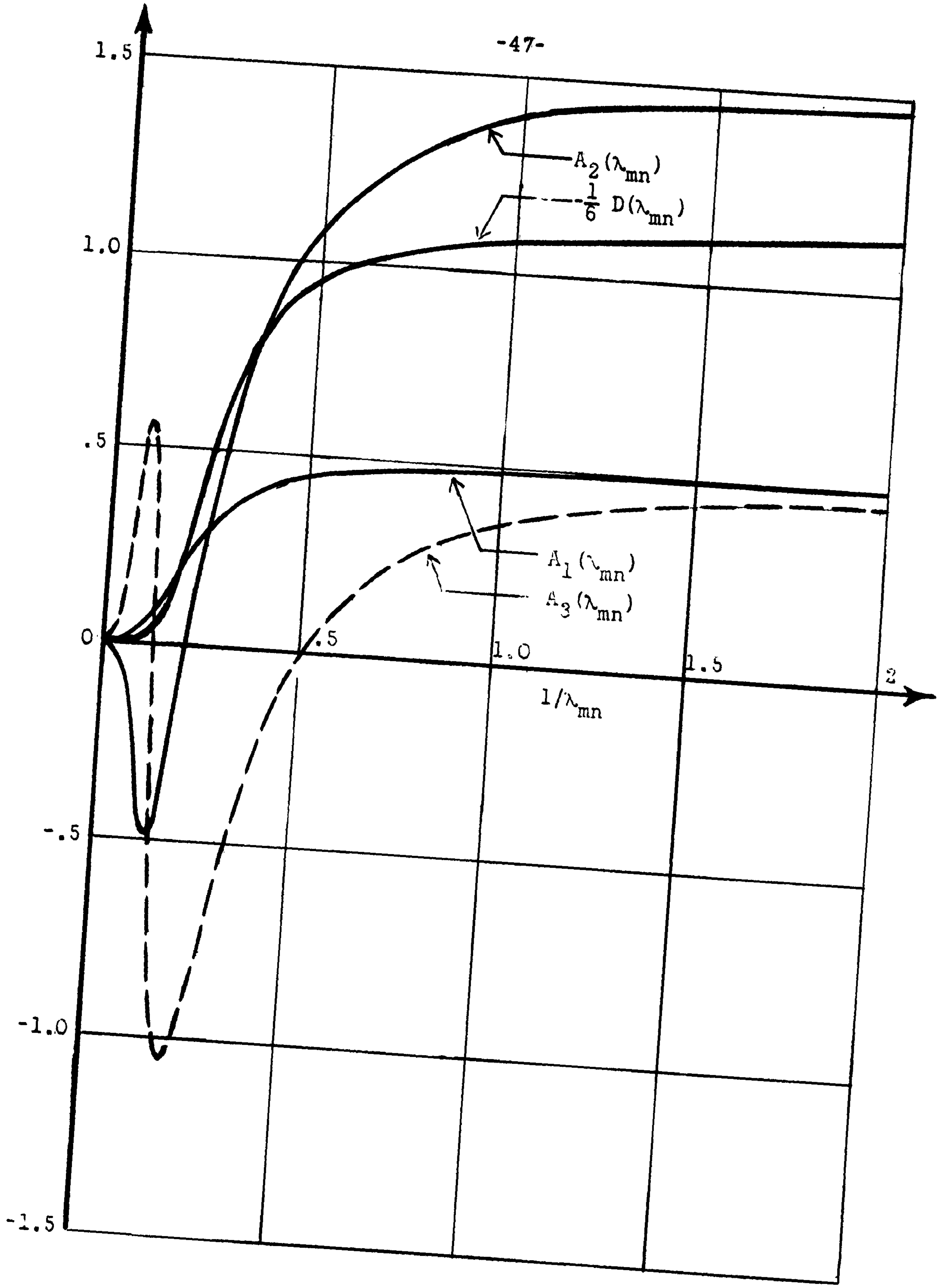

Fig. 2 Functions for Generalized Galerkin Method, $A_i(\lambda_{mn})$, $D(\lambda_{mn})$

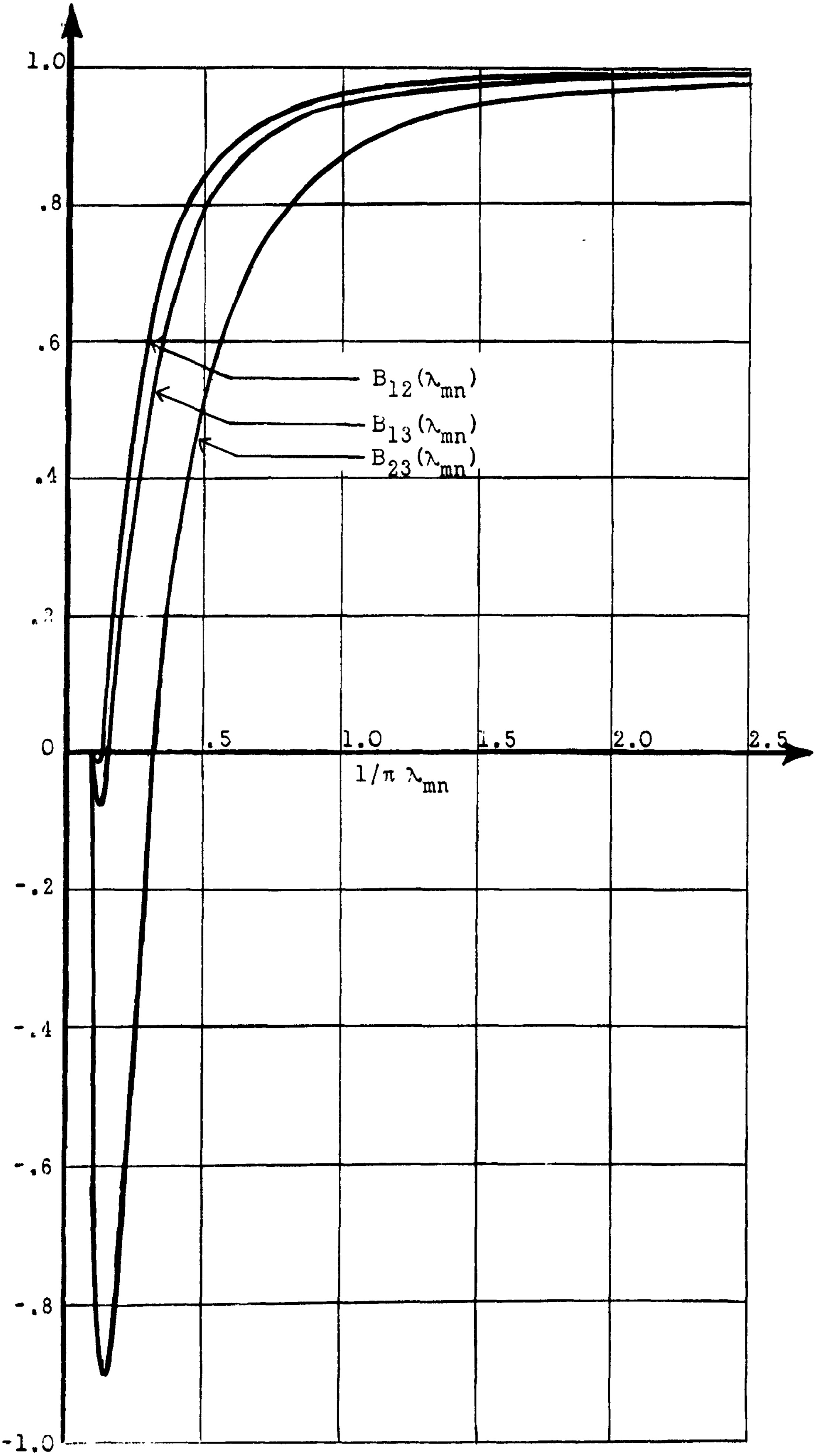

Fig. 3 Functions for Generalized Galerkin Method, $B_{ij}(\lambda_{mn})$

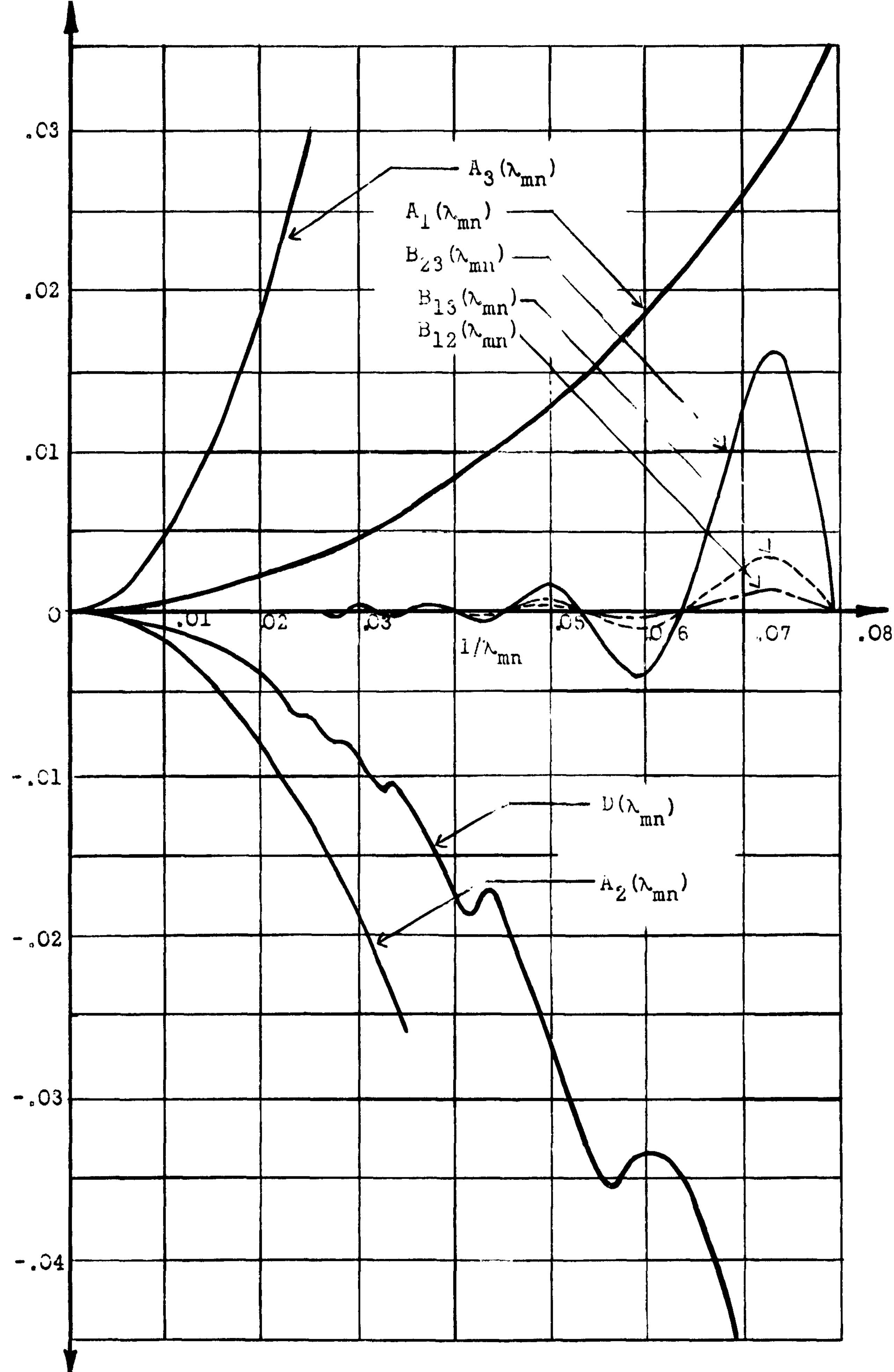

Fig. 4 Functions for the Generalized Galerkin Method, small values of. $A_i(\lambda_{mn})$, $D(\lambda_{mn})$, $B_{ij}(\lambda_{mn})$

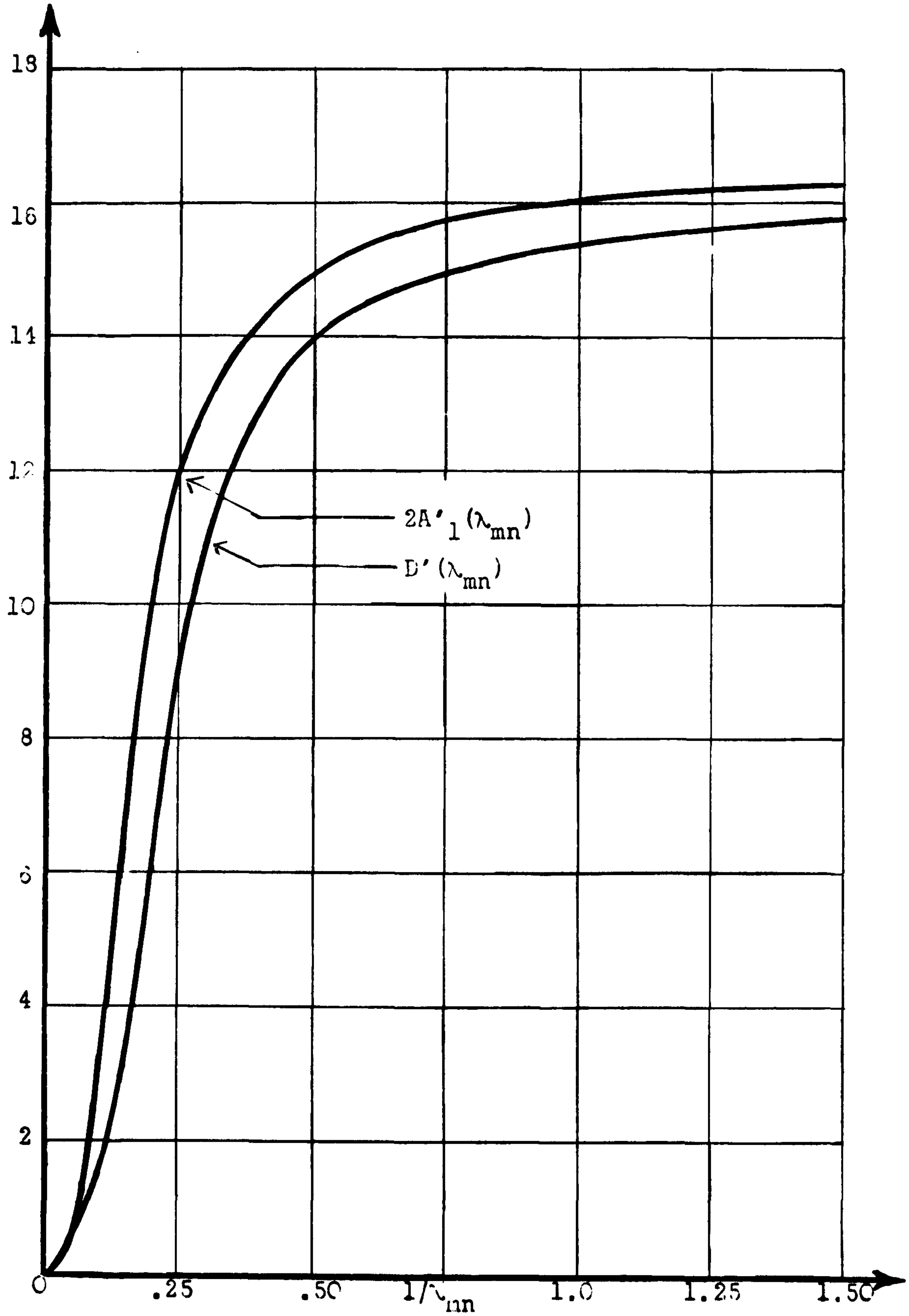

Fig. 5 Functions for Generalized Galerkin Method, $A_1'(\lambda_{mn})$, $D'(\lambda_{mn})$

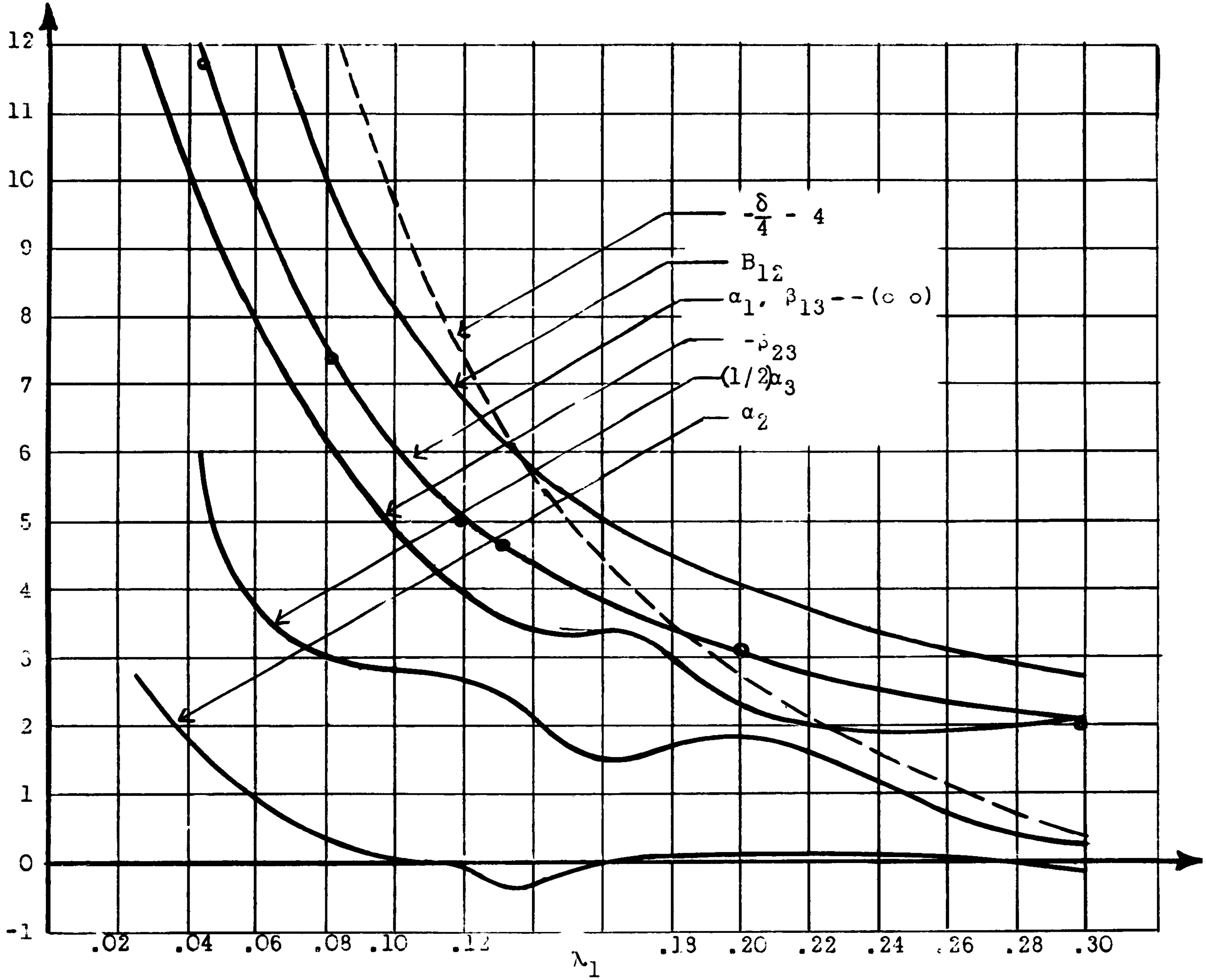

Fig. 6 Functions for Generalized Galerkin Method, α_i, δ, β_{ij}; central impact on simply supported circular plate

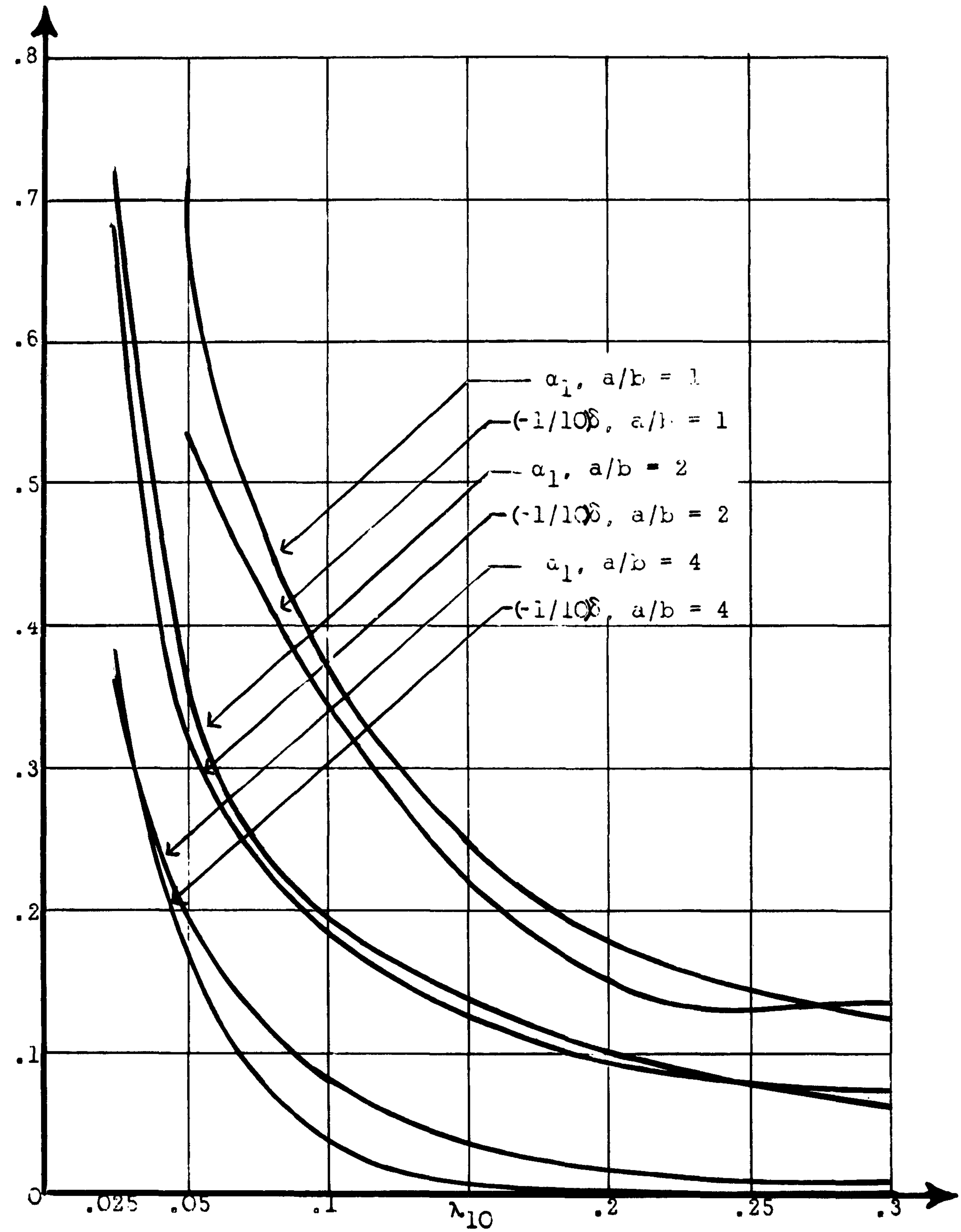

Fig. 7 Functions for Generalized Galerkin Method, α_1, δ, central impact on simply supported rectangular plate, (a/b=1, 2, 4)

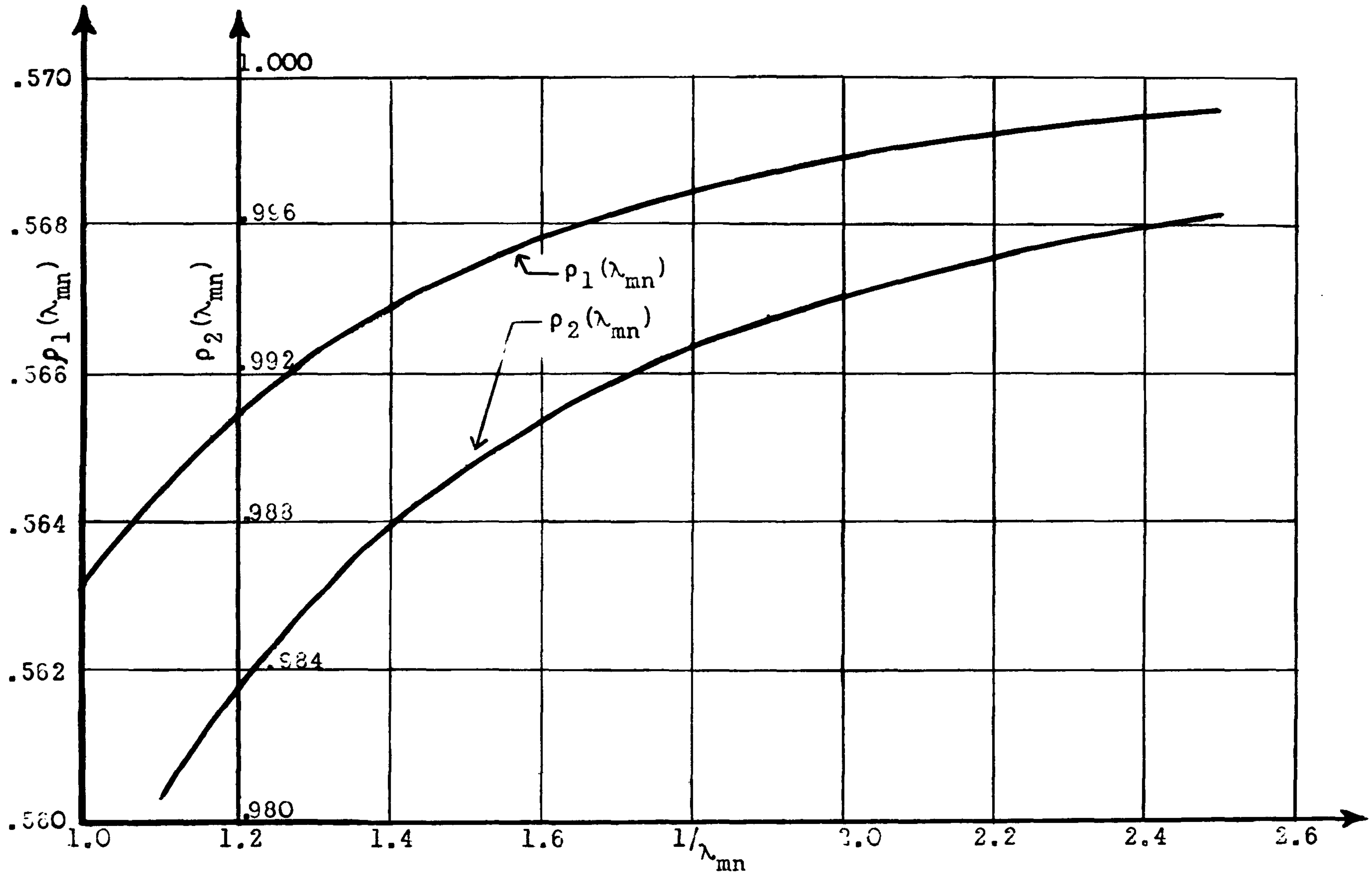

Fig. 8 Functions for the Collocation Method,

$\rho_1(\lambda_{mn})$, $\rho_2(\lambda_{mn})$; case (a), large values

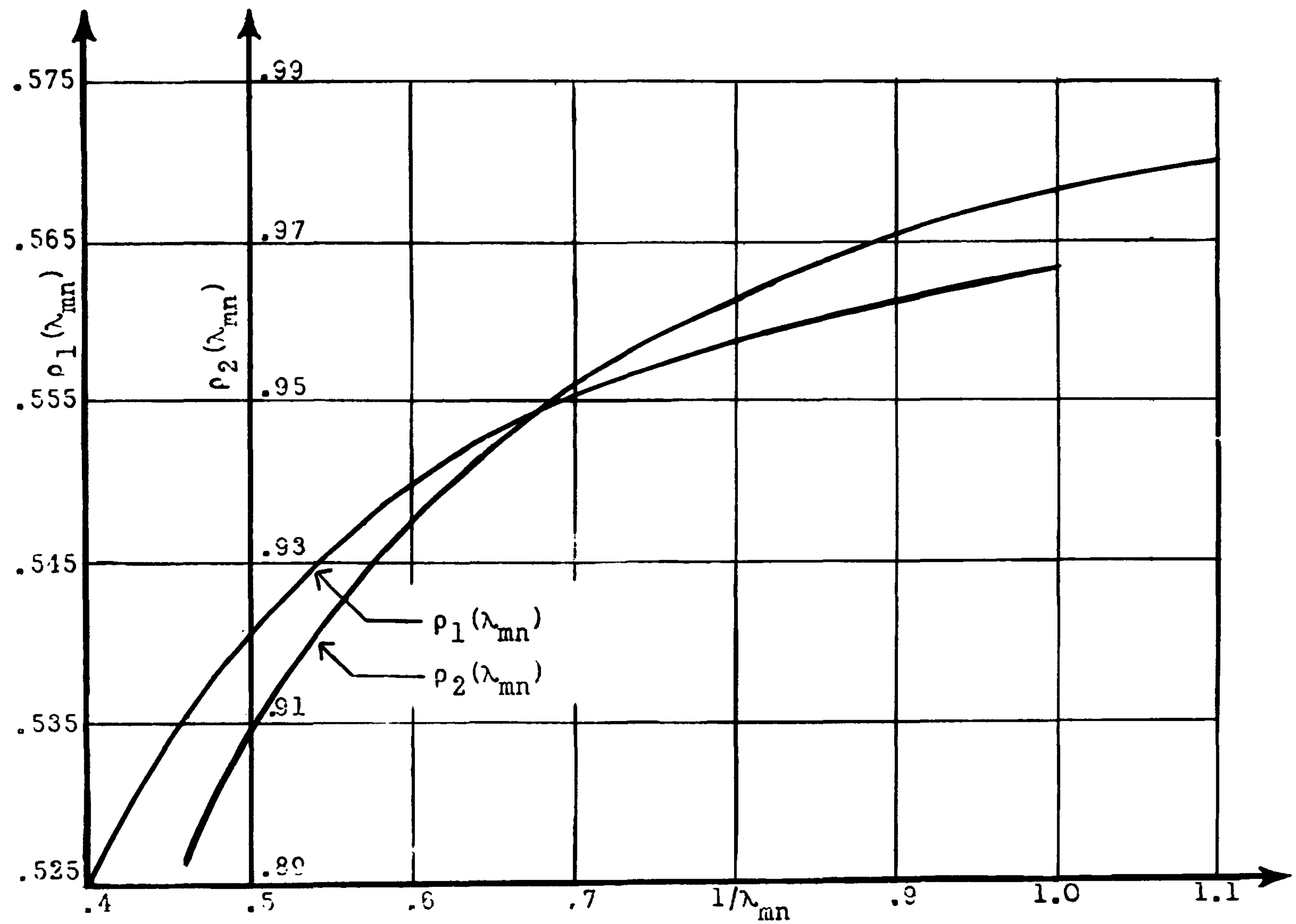

Fig. 9 Functions for the Collocation Method, $\rho_1(\lambda_{mn})$, $\rho_2(\lambda_{mn})$; case (a), smaller values

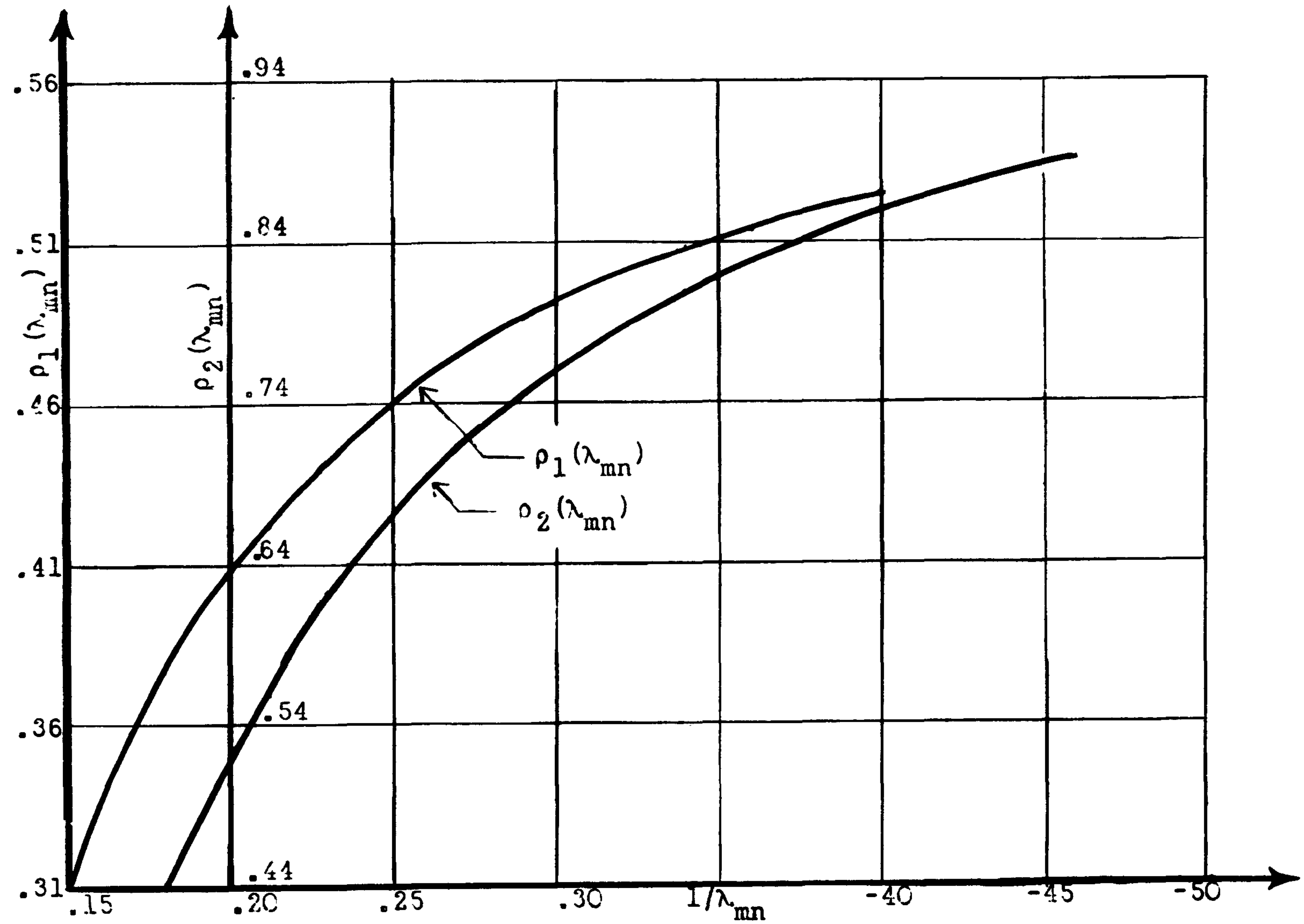

Fig. 10 Functions for the Collocation Method,

$\rho_1(\lambda_{mn})$, $\rho_2(\lambda_{mn})$; case (a), medium values

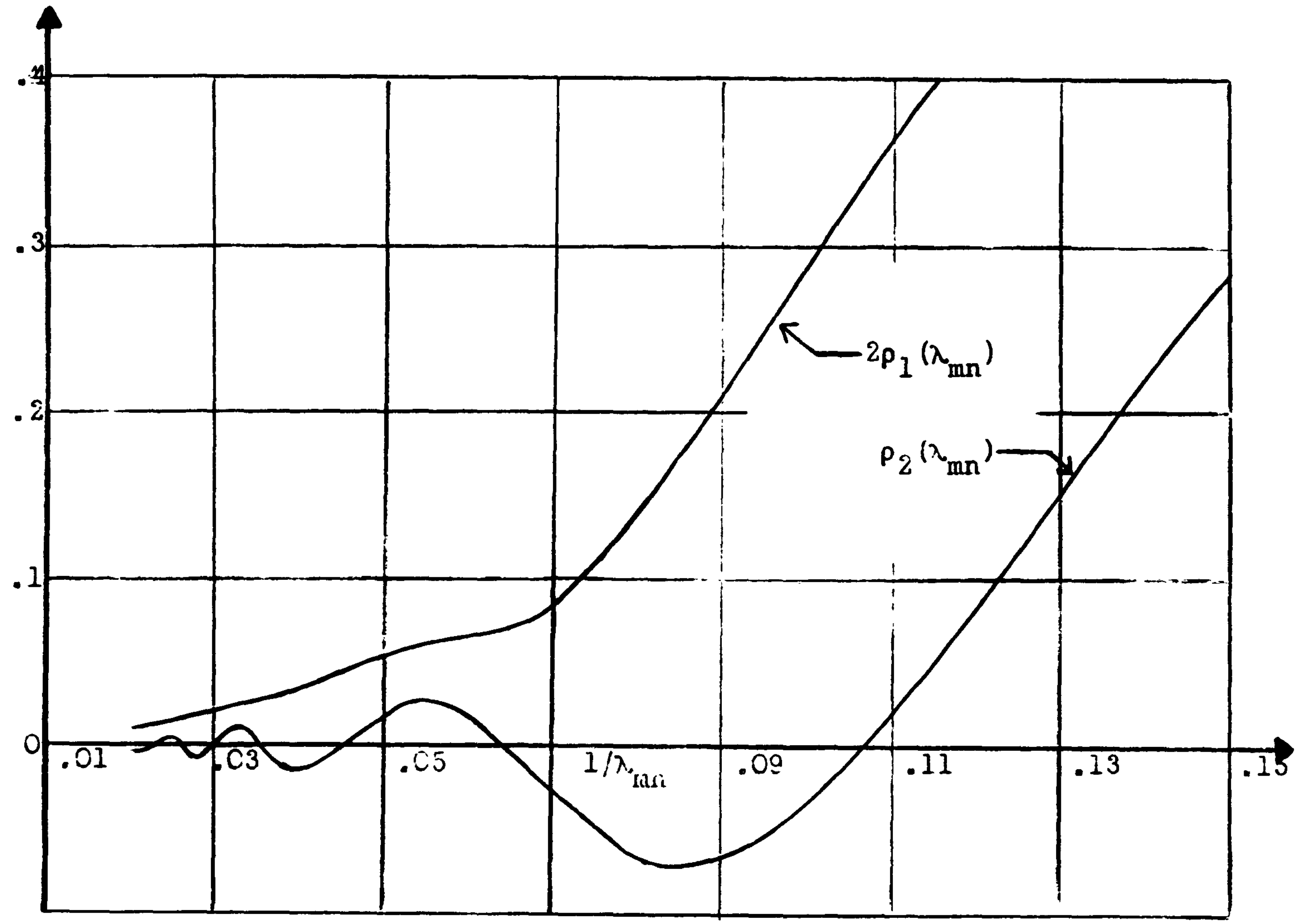

Fig. 11 Functions for the Collocation Method,
$\rho_1(\lambda_{mn})$, $\rho_2(\lambda_{mn})$; case (a), small values

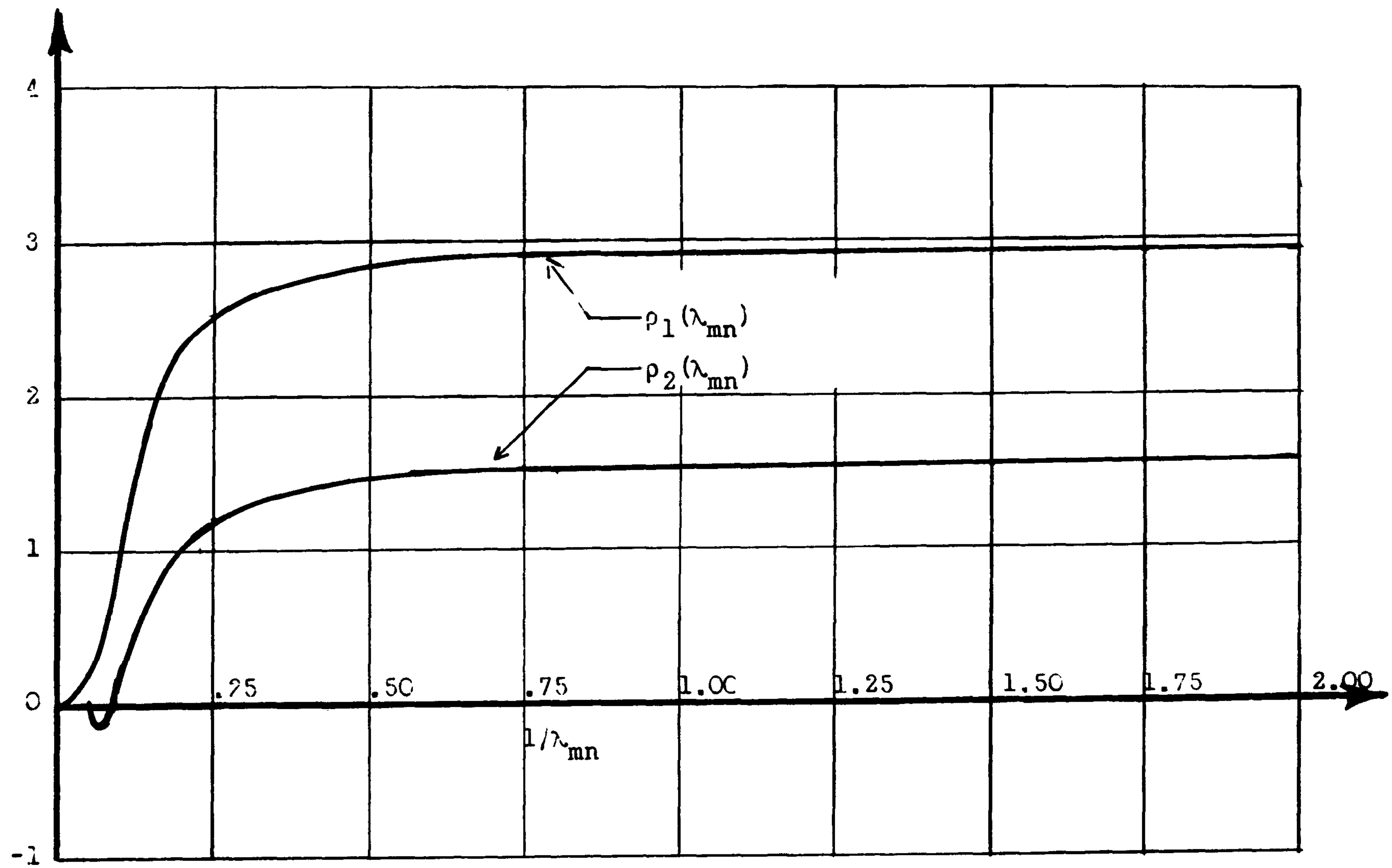

Fig. 12 Functions for the Collocation Method,

$\rho_1(\lambda_{mn})$, $\rho_2(\lambda_{mn})$; case (b)

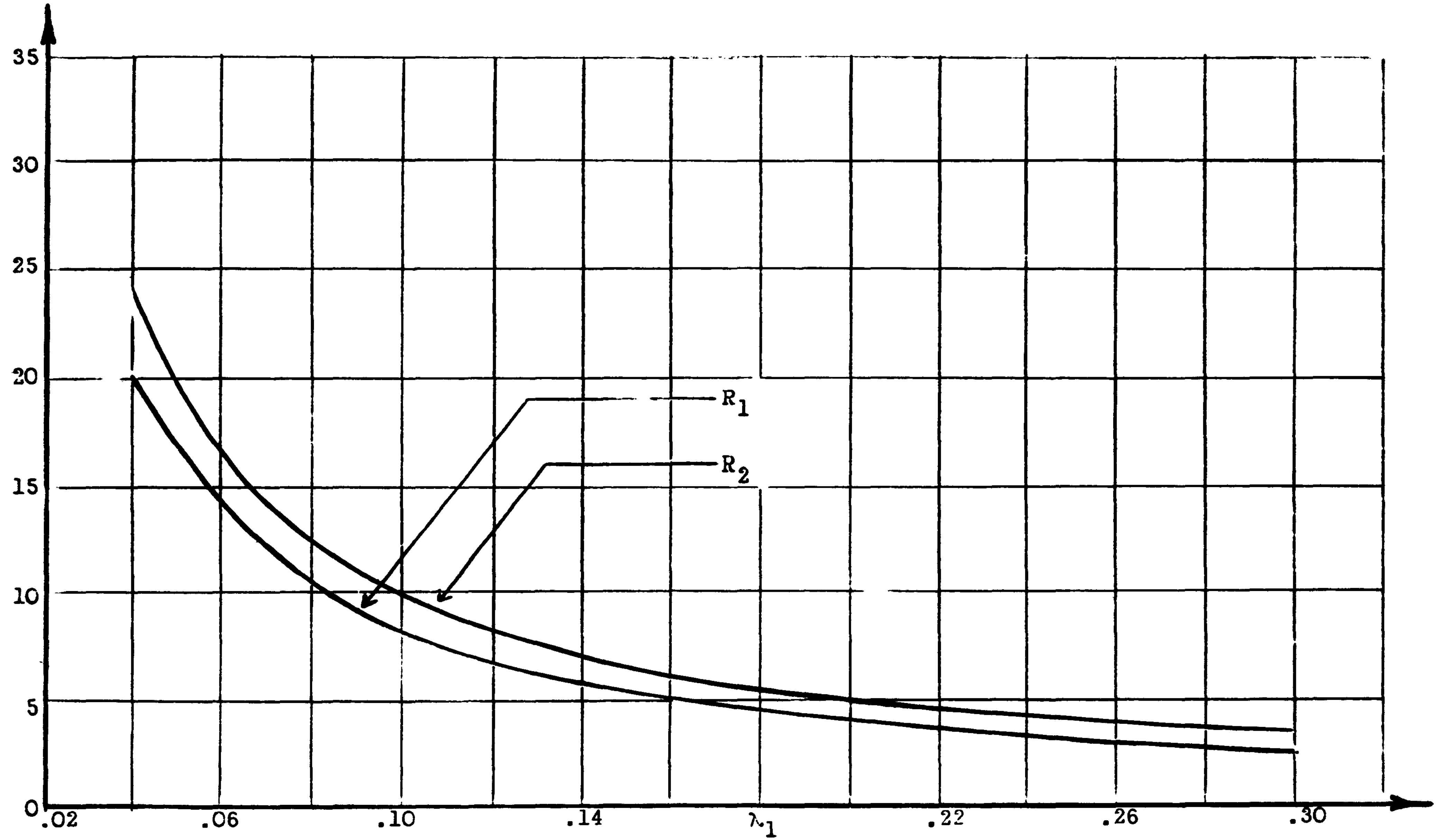

Fig. 13 Functions for the Collocation Method, R_1, R_2, central impact on simply supported circular plate

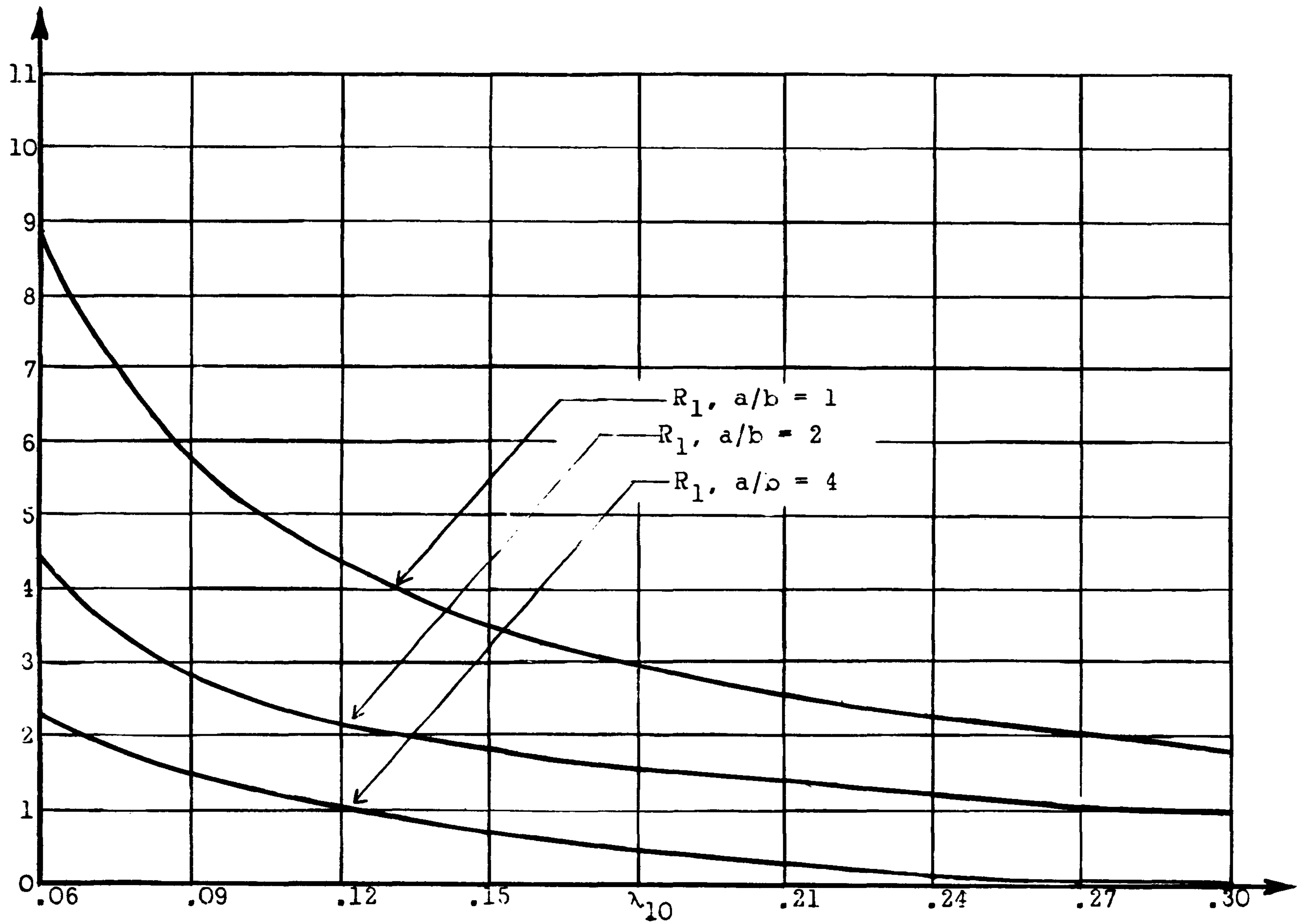

Fig. 14 Functions for the Collocation Method, R_1, central impact on simply supported rectangular plate, (a/b=1,2,4)

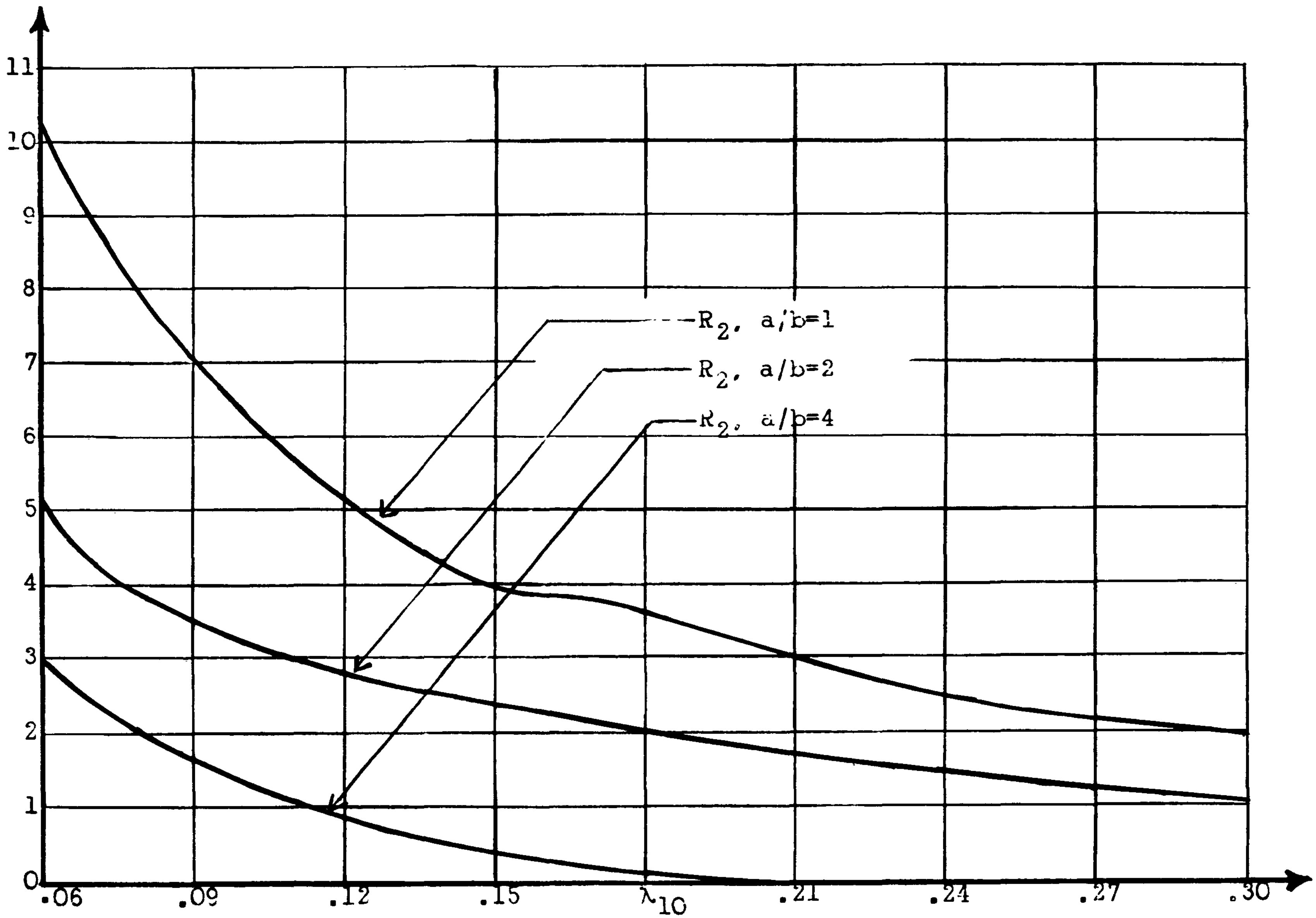

Fig. 15 Functions for the Collocation Method, R_2, central impact on simply supported rectangular plate, (a/b=1,2,4)

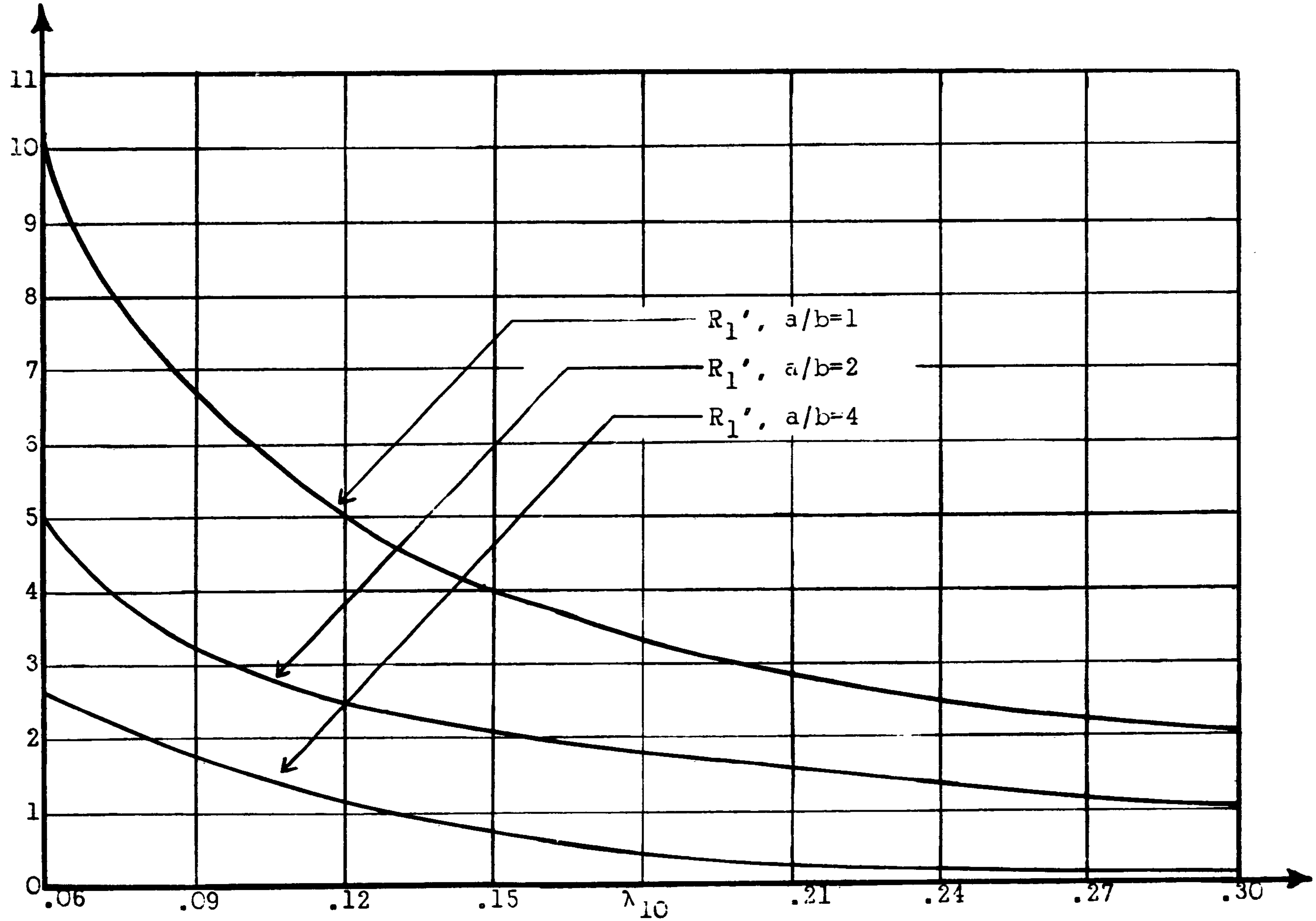

Fig. 16 Functions for the Collocation Method, R_1', central impact on simply supported rectangular plate, (a/b=1,2,4)

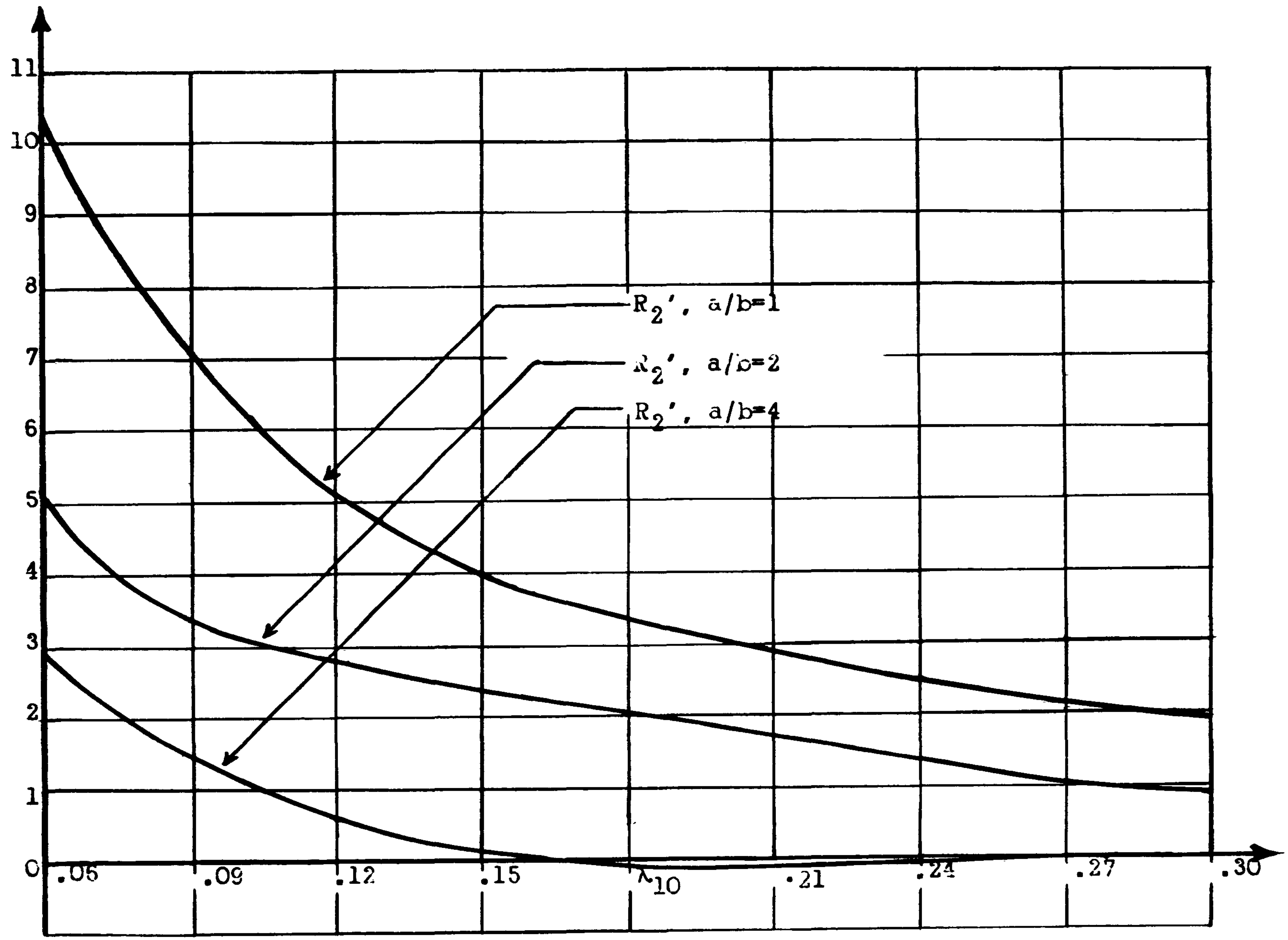

Fig. 17 Functions for the Collocation Method, R_2'.
central impact on simply supported rectangular plate, (a/b=1,2,4)

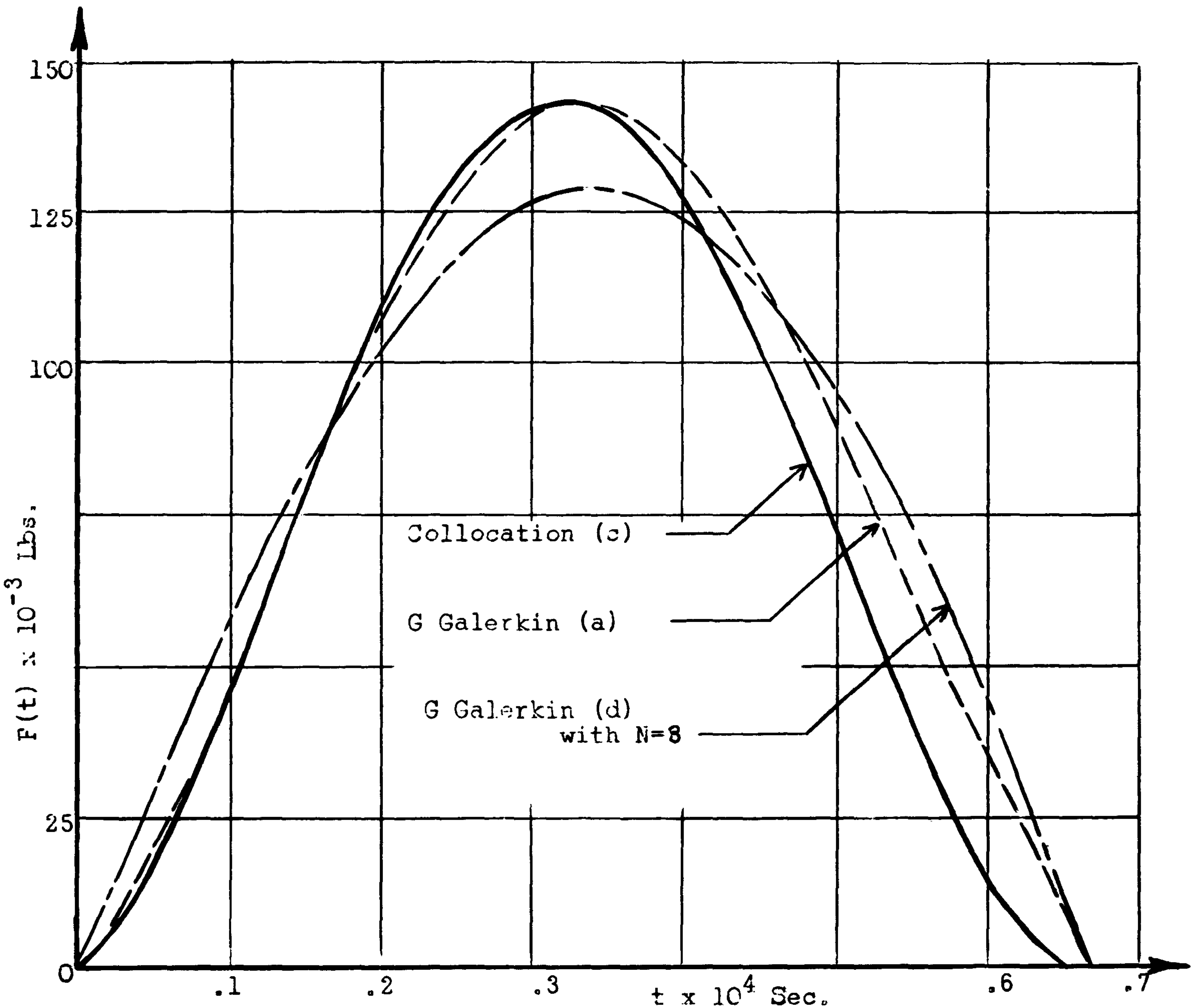

Fig. 18 G Galerkin, Collocation; Force-time curves for central impact on a simply supported circular plate

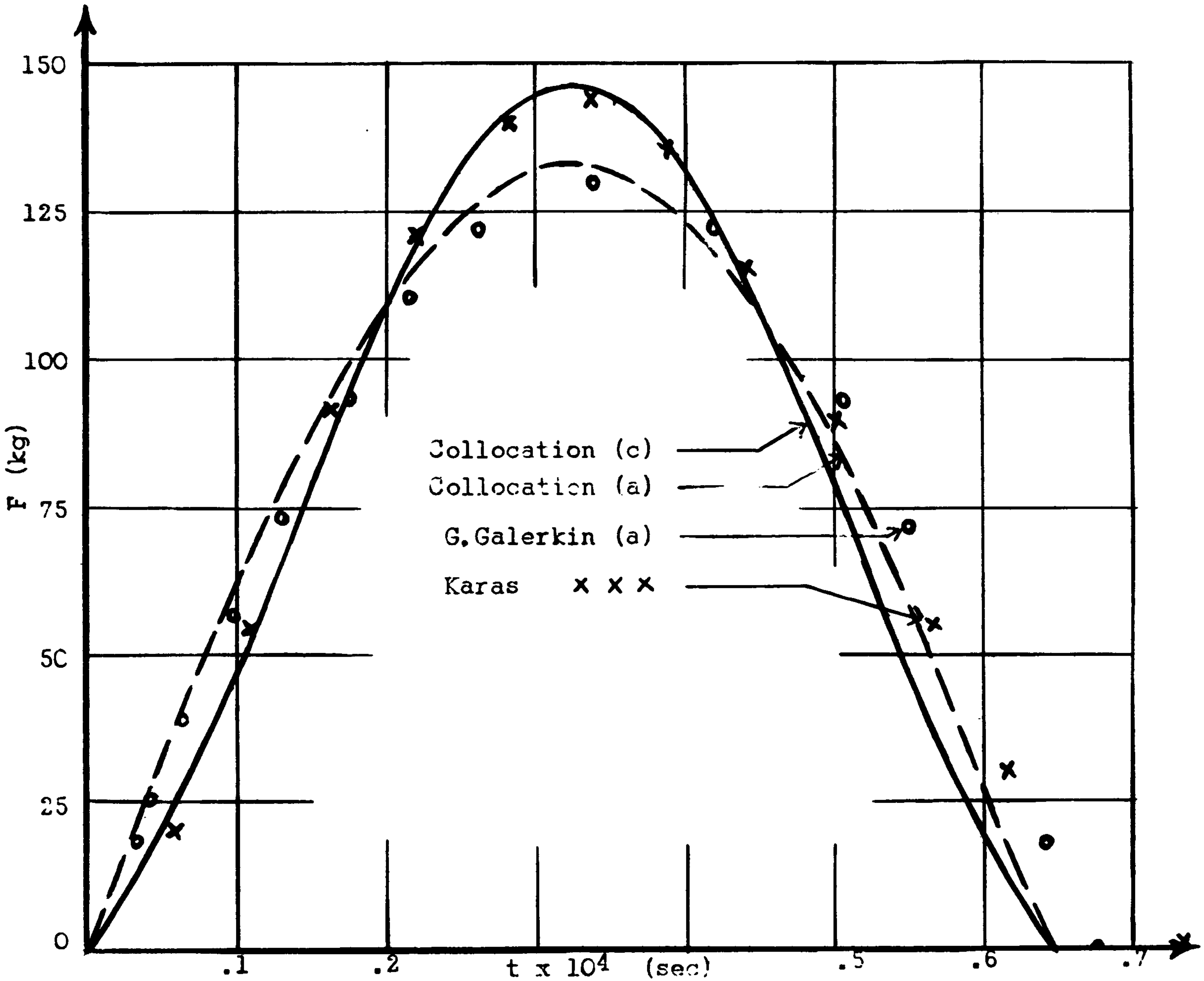

Fig. 19 G.Galerkin, Collocation (a), (c); Force-time curves for central impact on simply supported square plate

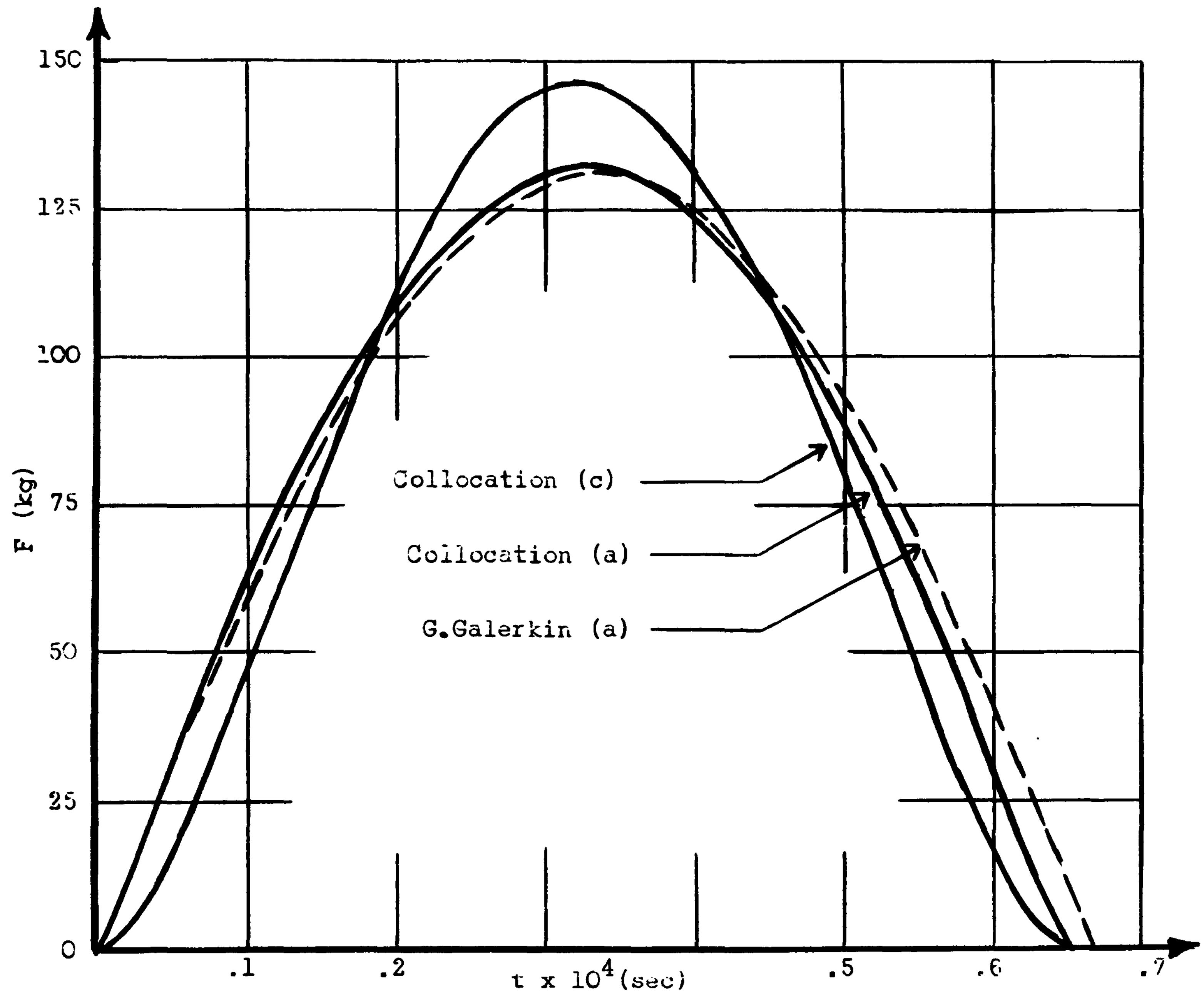

Fig. 20 G.Galerkin, Collocation (a),(c); Force-time curves for central impact on a simply supported rectangular plate, (a/b = 2)

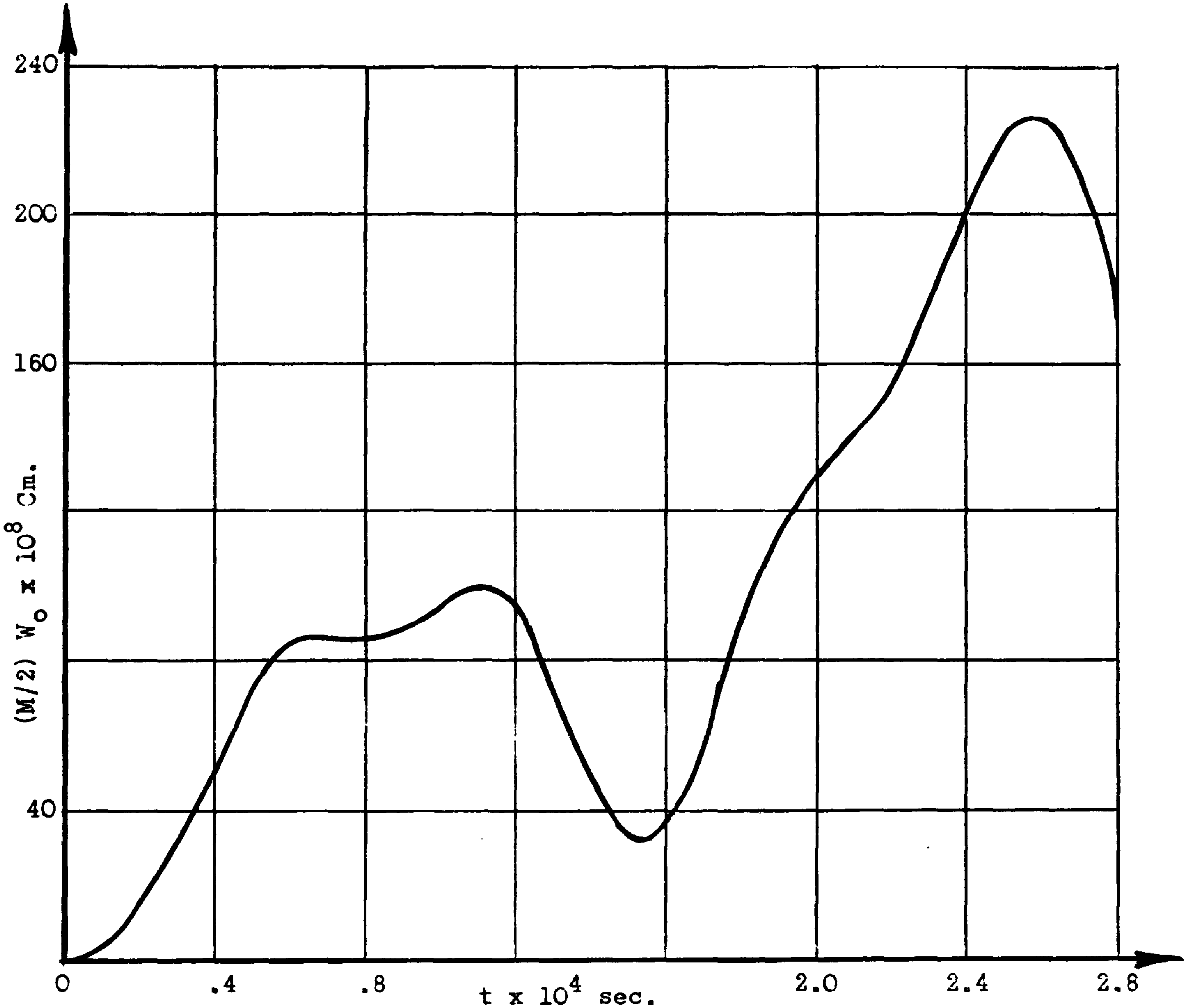

Fig. 21 Deflection curve, circular plate

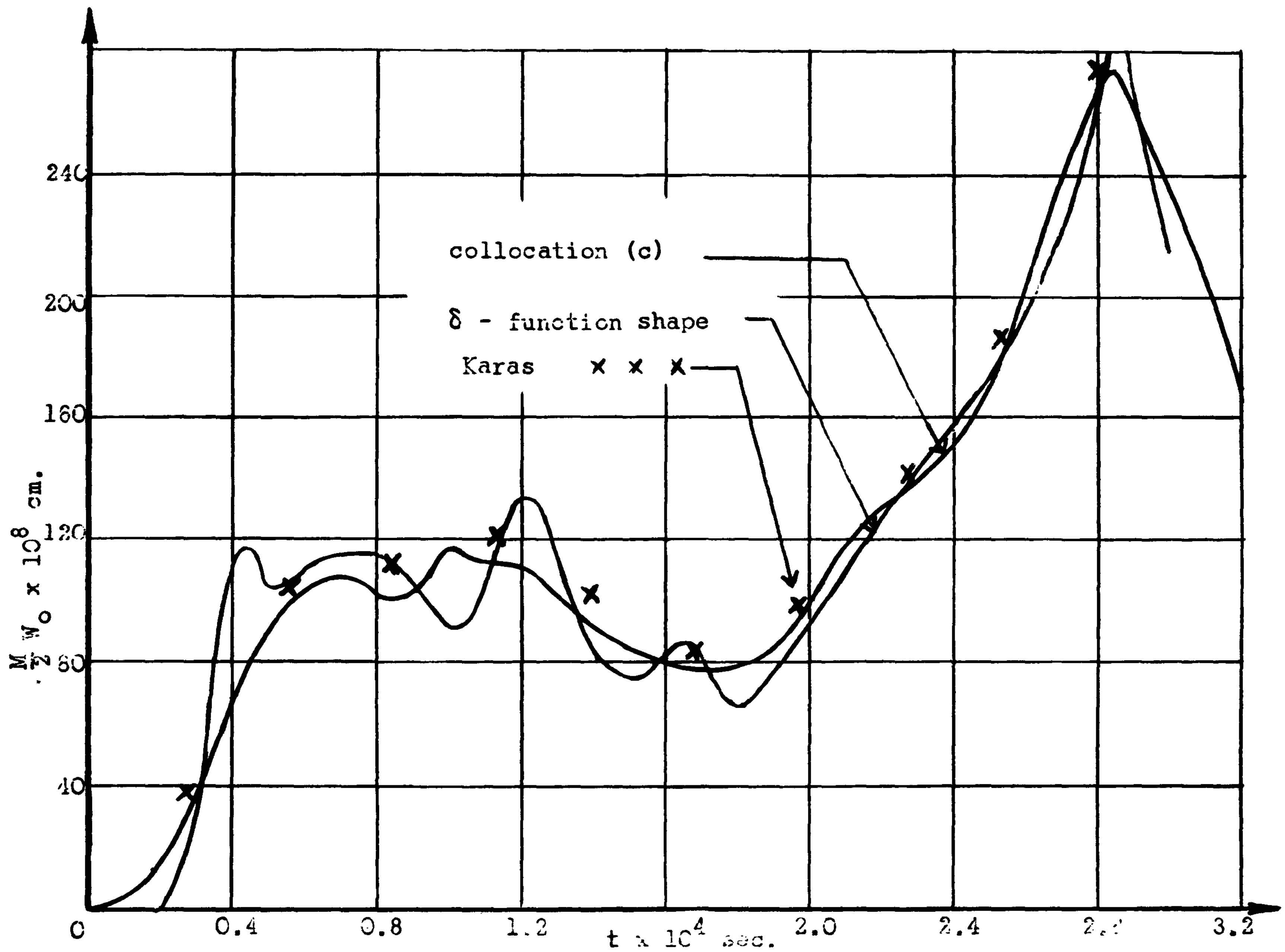

Fig. 22 Deflection curves, square plate

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